PHYS 610 Mathematical Methods for Scientists F 2020

Problem Assignment # 3 10/15/2020
due 10/22/2020

1.2.2 **Products**

Prove the corollary to proposition 2 of ch.1 §2.2: If a is an element of a multiplicative group, and $n, m \in \mathbb{N}$, then

a) $a^n\, a^m = a^{n+m}$

b) $(a^n)^m = a^{nm}$

(2 points)

1.2.3 **The group S_3**

a) Compile the group table for the symmetric group S_3. Is S_3 abelian?

b) Find all subgroups of S_3. Which of these are abelian?

(6 points)

1.2.4 **Abelian groups**

Let $(G, \vee)$ be a group with neutral element e. Let $a \in G$ be a fixed element, and define a mapping $\varphi : G \to G$ by $\varphi(x) = a \vee x \vee a^{-1}\ \forall x \in G$.

a) Show that φ defines an automorphism on G, called an *inner automorphism*.

b) Show that abelian groups have no inner automorphisms except for the identity mapping $\varphi(x) = x$.

c) Let $g \vee g = e\ \forall g \in G$. Prove that G is abelian.

(6 points)

1.3.1 **Fields**

a) Show that the set of rational numbers $\mathbb{Q}$ forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set F with two elements, $F = \{\theta, e\}$. On F, define an operation "plus" $(+)$, about which we assume nothing but the defining properties

$$\theta + \theta = \theta \quad , \quad \theta + e = e + \theta = e \quad , \quad e + e = \theta$$

Further, define a second operation "times" $(\cdot)$, about which we assume nothing but the defining properties

$$\theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta \quad , \quad e \cdot e = e$$

Show that with these definitions (and **no** additional assumptions), F is a field.

(7 points)

1.2.2.) a) We want to show that $a^n a^m = a^{n+m}$.

let $m=1$: Then $a^n a = \left(\prod_{\nu=1}^{n} a\right) a = \prod_{\nu=1}^{n+1} a = a^{n+1}$

by the recursive definition.

$m \to m+1$: $a^n a^{m+1} = a^n a^m a = a^{n+m} a = a^{n+m+1}$

1 $\rightarrow$ The statement holds $\forall m \in \mathbb{N}$ by induction.

b) We want to show that $(a^n)^m = a^{nm}$.

let $m=1$: $(a^n)^1 = a^n = a^{n\cdot 1}$ ✓

$m \to m+1$: $(a^n)^{m+1} = (a^n)^m a^n = a^{nm} a^n = a^{nm+n}$

$= a^{n(m+1)}$

1 $\rightarrow$ The statement holds $\forall m \in \mathbb{N}$ by induction.

1.2.3-10) The elements of S_3 are

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

1 $$P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

With this representation, the group table is

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_5	P_6	P_3	P_4
P_3	P_3	P_4	P_1	P_2	P_6	P_5
P_4	P_4	P_3	P_6	P_5	P_1	P_2
P_5	P_5	P_6	P_2	P_1	P_4	P_3
P_6	P_6	P_5	P_4	P_3	P_2	P_1

1 S_3 is not abelian: E.g., $P_2 \circ P_3 = P_5$, $P_3 \circ P_2 = P_4$.

b) Consider the group table from problem 9.) . Now consider subsets of S_3 that contain

<u>5 elements</u>: $\{P_2, P_3, P_4, P_5, P_6\}$ does not contain $P_1 = E$

$\{P_1, P_3, P_4, P_5, P_6\}$ is not closed, since $P_3 \circ P_4 = P_2$

(1) same for the other 4 possibilities

<u>4 elements</u>: The subset must contain P_1 -> We can form

$\{P_1, P_2, P_3, P_4\}$ not closed since $P_2 \circ P_3 = P_5$

$\{P_1, P_2, P_3, P_5\}$ " since $P_3 \circ P_2 = P_4$

$\{P_1, P_2, P_4, P_5\}$ " since $P_4 \circ P_2 = P_3$

$\{P_1, P_3, P_4, P_5\}$ " since $P_3 \circ P_4 = P_2$

(1) same for the other 6 possibilities

<u>3 elements</u>: Consider $\{P_1, P_4, P_5\}$, which has a group table

	P_1	P_4	P_5
P_1	P_1	P_4	P_5
P_4	P_4	P_5	P_1
P_5	P_5	P_1	P_4

This is an abelian subgroup!

Whereas,

$\{P_1, P_2, P_3\}$ is not closed since $P_2 \circ P_3 = P_5$

and the same for the other 8 possibilities.

(1)

2 elements : $\{P_1, P_2\}$ is an abelian subgroup

$\{P_1, P_3\}$ "

$\{P_1, P_4\}$ is not closed

$\{P_1, P_5\}$ "

$\{P_1, P_6\}$ is an abelian subgroup

(1)

1 element : $\{P_1\}$ trivially is an abelian subgroup

$\Rightarrow$ The subgroups of S_3 are:

$S_3^{(1)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$
$S_3^{(2)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$
$S_3^{(3)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$
$S_3^{(4)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$

They are all abelian

1.2.4.) a) We need to show that $\varphi(x) = a \vee x \vee a^{-1}$ is bijective.

First show that φ is surjective: Let $y \in G$. Then $x = a^{-1} \vee y \vee a$

1 gets mapped onto y, since $\varphi(x) = a \vee a^{-1} \vee y \vee a \vee a^{-1} = y$.

Now show that φ is injective. Let $\varphi(x_1) = \varphi(x_2)$

$\rightsquigarrow a \vee x_1 \vee a^{-1} = a \vee x_2 \vee a^{-1}$

$\rightsquigarrow a^{-1} \vee a \vee x_1 \vee a^{-1} \vee a = a^{-1} \vee a \vee x_2 \vee a^{-1} \vee a \rightsquigarrow x_1 = x_2$

1 $\rightsquigarrow$ φ is bijective

Now show that φ respects the operation $\vee$:

$$\varphi(x) \vee \varphi(y) = a \vee x \vee \underbrace{a^{-1} \vee a}_{=e} \vee y \vee a^{-1} = a \vee x \vee e \vee y \vee a^{-1} = a \vee x \vee y \vee a^{-1} = \varphi(x \vee y) \checkmark$$

1 $\rightsquigarrow$ φ is an automorphism

b) Let G be abelian. $\rightsquigarrow \varphi(x) = a \vee x \vee a^{-1} = x \vee a \vee a^{-1} = x \vee e = x$

1 $\rightsquigarrow$ If φ is an inner automorphism, then φ is the identity.

c) We need to show that $g_1 \vee g_2 = g_2 \vee g_1 \quad \forall g_1, g_2 \in G$

We know that

$$e = g \vee g \quad \forall g \in G$$

$\rightsquigarrow$ this holds in particular for $g = g_1 \vee g_2 \in G$ and also for $g = g_2$

$$\rightsquigarrow \quad g_2 \vee g_2 = e = (g_1 \vee g_2) \vee (g_1 \vee g_2)$$

$$= g_1 \vee g_2 \vee g_1 \vee g_2 \quad \text{by associativity}$$

1

$$\rightsquigarrow \quad g_2 \vee \underbrace{g_2 \vee g_2}_{=e} = g_1 \vee g_2 \vee g_1 \vee \underbrace{g_2 \vee g_2}_{=e}$$

$$\rightsquigarrow \quad g_2 = g_1 \vee g_2 \vee g_1$$

$$\rightsquigarrow \quad g_1 \vee g_2 = \underbrace{g_1 \vee g_1}_{=e} \vee g_2 \vee g_1$$

1

$$\rightsquigarrow \quad \underline{\underline{g_1 \vee g_2 = g_2 \vee g_1}} \qquad \square$$

1.3.1.) a) $\mathbb{Q}$ is a group under addition with neutral element $0 \in \mathbb{Q}$:

(i) $q_1 + q_2 \in \mathbb{Q} \ \forall\, q_1, q_2 \in \mathbb{Q}$

(ii) Addition is associative and commutative

(iii) The number zero is an element of $\mathbb{Q}$, and $0 + q = q \ \forall q \in \mathbb{Q}$

1 (iv) Let $q \in \mathbb{Q}$. Then $\exists\, -q$: $q + (-q) = 0$

$\mathbb{Q} \setminus \{0\}$ is also a group under multiplication:

(i) $q_1 q_2 \in \mathbb{Q} \ \forall\, q_1, q_2 \in \mathbb{Q}$

(ii) Multiplication is associative and commutative

(iii) The number 1 is an element of $\mathbb{Q}$, and $1 \cdot q = q \ \forall\, q \in \mathbb{Q}$

(iv) Let $q \in \mathbb{Q}$ and $q \neq 0$. Then $\exists\, q^{-1} \equiv \frac{1}{q}$: $q q^{-1} = 1$.

1 Finally, ordinary addition and multiplication on $\mathbb{Q}$ are distributive.

$\rightarrow$ $\mathbb{Q}$ is a commutative field b.) We need to show that F is a group under addition.

(i) $a+b \in F \ \forall a,b \in F$ by definition → closure ✓

(ii) $(d+e)+d = e+d = e = d+(e+d)$

$(e+e)+d = d+d = d = e+(e+d)$

1 → "+" is associative

(iii) d is the neutral element by definition.

(iv) $-d=d$, $-e=e$ by definition → existence of inverse ✓

(v) "+" is commutative by definition

1 → F is an abelian group under "+".

We also need to show that $F \setminus \{d\}$ is a group under "$\cdot$".

But $F\setminus\{d\} = \{e\}$, and

(i) closure ✓ by definition

(ii) associativity is trivial

(iii) e is neutral element by definition

(iv) e is its own inverse

1 → $F\setminus\{d\}$ is a group under "$\cdot$". It is trivially abelian.

Finally, we must check the distribution laws. Since "$\cdot$" is abelian we only have to show that $(a+b)\cdot c = a\cdot c + b\cdot c \ \forall a,b,c \in F$. (*)

1 (i) $c=d$. → $(a+b)\cdot d = d = a\cdot d + b\cdot d$ irrespective of a,b ✓

(ii) $c=e$. If either $a=d$ or $b=d$, (*) holds.

If $a=b=e$, $(e+e)\cdot e = d\cdot e = d$

1 and $e\cdot e + e\cdot e = d+d = d$ → distribution law ✓

→ F is a field