PHYS 610 Mathematical Methods for Scientists F 2020

Problem Assignment # 4 10/22/2020
due 10/29/2020

1.4.1. **Function space**

Consider the set C of continuous functions $f : [0,1] \to \mathbb{R}$. Show that by suitably defining an addition on C, and a multiplication with real numbers, one can make C an additive vector space over $\mathbb{R}$.

(2 points)

1.4.2. **The space of rank-2 tensors**

a) Prove the theorem of ch.1 §4.3: Let V be a vector space V of dimension n over K. Then the space of rank-2 tensors, defined via bilinear forms $f : V \times V \to K$, forms a vector space of dimension n^2.

b) Consider the space of bilinear forms f on V that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

hint: On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

1.4.3. **Cross product of 3-vectors**

Let $x, y \in \mathbb{R}_3$ be vectors, and let ϵ_{ijk} be the Levi-Civita symbol. Show that the (covariant) components of the cross product $x \times y$ are given by

$$(x \times y)_i = \epsilon_{ijk} x^j y^k$$

(1 point)

1.4.4. **Symmetric tensors**

Let V be an n-dimensional vector space over K with some basis, let $f : V \times V \to K$ be a bilinear form, and let t be the rank-2 tensor defined by f. Show that f is symmetric, i.e. $f(x,y) = f(y,x)\ \forall x, y \in V$, if and only if the components of the tensor with respect to the given basis are symmetric, i.e., $t_{ij} = t_{ji}$.

(2 points)

1.4.1.) On C, define $(f+g)(x) := f(x) + g(x)$

If f and g are continuous, then so is the so defined $(f+g)$

$\leadsto$ closure ✓

Furthermore, since $f(x) \in \mathbb{R}$, C inherits all of the other group properties from $(\mathbb{R}, +)$

1 $\leadsto$ C is an additive group

Now define multiplication with scalars $\lambda \in \mathbb{R}$ by $(\lambda f)(x) := \lambda f(x)$

If f is continuous, then so is the so defined (λf).

Furthermore, since $\lambda \in \mathbb{R}$ and $f(x) \in \mathbb{R}$, this multiplication with scalars is bilinear and associative, as it inherits these properties from $\mathbb{R}$ under ordinary addition and multiplication of numbers

Finally, $(1f)(x) = 1 f(x) = f(x) \quad \forall x \in [0,1] \leadsto 1f = f$

1 $\leadsto$ C is an $\mathbb{R}$-vector space

$\square$

1.4.2 a) We know that the rank-2 tensors are one-to-one correspondent to bilinear forms $f(x,y)$. On the set of bilinear forms, define an addition by

$$(f+g)(x,y) := f(x,y) + g(x,y)$$

This makes the set of forms an additive group.
Define a multiplication with scalars by

$$(\lambda f)(x,y) := \lambda f(x,y) \quad , \quad \lambda \in k$$

This makes the space of forms a k-vector space.
On the space of rank-2 tensors $t, u, \dots$ this corresponds to defining the tensor $t+u$ as the tensor with coordinates

1

$$(t+u)_{ij} = t_{ij} + u_{ij}$$

and the tensor λt as the one with coordinates

$$(\lambda t)_{ij} = \lambda t_{ij}$$

1

The space of tensors is now a k-vector space

Consider a basis $\{e_i\}$ of V, and construct n^2 tensors

$$E_{ij} := e_i \otimes e_j$$

with (contravariant) coordinates

$$(E_{ij})^{kl} = \delta_i^{\,k} \delta_j^{\,l}$$

Define a tensor t as a linear combination of the E_{ij},

$$t = \sum_{ij} \tau^{ij} E_{ij} \quad \text{with coefficients } \tau^{ij} \in k$$

This tensor has coordinates

$$t^{kl} = \sum_{ij} \tau^{ij} (E_{ij})^{kl} = \tau^{kl}$$

→ Any rank-2 tensor can be written as a linear combination of the E_{ij}, with the coordinates t^{ij} of t as the coefficients:

$$t = \sum_{ij} t^{ij} E_{ij}$$

1 → The E_{ij} span the space

But in order for t to be the null tensor, all of its coordinates must be zero, so $t=0$ implies $t^{ij}=0 \ \forall i,j$

→ The E_{ij} are linearly independent

1 → The n^2 rank-2 tensors E_{ij} form a basis of the space of rank-2 tensors, and hence the space has dimension n^2.

b) Let f_{ij} be the bilinear form that corresponds to the tensor E_{ij}. Then

$$f_{ij}(e_k, e_l) = (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \quad \text{with } \delta_{ik} \text{ the Kronecker symbol}$$

For arbitrary $x, y \in V$ we have

$$f_{ij}(x,y) = x^k y^l f(e_k, e_l) = x^k y^l \delta_{ik}\delta_{jl} = x^i y^j$$

1 → The set of n^2 bilinear forms f_{ij} defined by

$$\boxed{f_{ij}(x,y) = x^i y^j}$$

forms a basis of the space of bilinear forms.

1.4.3.) Let $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$. The cross product is defined by

$$x \times y = (x^2 y^3 - x^3 y^2, x^3 y^1 - x^1 y^3, x^1 y^2 - x^2 y^1)$$

On the other hand,

$$\underline{\underline{\varepsilon_{ijk} x^j y^k}} = \begin{cases} x^2 y^3 - x^3 y^2 & \text{if } i = 1 \\ x^3 y^1 - x^1 y^3 & \text{if } i = 2 \\ x^1 y^2 - x^2 y^1 & \text{if } i = 3 \end{cases} = \underline{\underline{(x \times y)^i}}$$

1

1.4.4.) Let $f(x,y) = f(y,x)$ $\forall x,y \in V$.

The tensor components are defined by $t_{ij} = f(e_i, e_j)$

$\leadsto$ $t_{ji} = f(e_j, e_i) = f(e_i, e_j) = t_{ij}$

(1)

Now let $t_{ij} = t_{ji}$

$\leadsto$ $f(e_i, e_j) = f(e_j, e_i)$ for all basis vectors e_i.

But for any $x,y \in V$ we have

$f(x,y) = x^i y^j f(e_i, e_j) = x^i y^j f(e_j, e_i) = y^j x^i f(e_j, e_i) = f(y,x)$

(1)

□