PHYS 610 Mathematical Methods for Scientists S 2020

Problem Assignment # 5 10/29/2020
due 11/05/2020

1.4.5. $\mathbb{R}$ **as a metric space**

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

1.4.6. **Limits of sequences**

a) Show that a sequence in a metric space has at most one limit.

hint: Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequency with a limit is a Cauchy sequence.

(3 points)

1.4.7. **Banach space**

Let B be a K-vector space (k = $\mathbb{R}$ or $\mathbb{C}$) with null vector θ. Let $||\ldots|| \ : \ \mathrm{B} \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0 \ \forall \ x \in \mathrm{B}$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y|| \ \forall \ x, y \in \mathrm{B}$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \ \forall \ x \in \mathrm{B}, \lambda \in \mathrm{K}$.

Then we call $||\ldots||$ a **norm** on B, and $||x||$ the **norm** of x.

Further define a mapping $d \ : \ \mathrm{B} \times \mathrm{B} \to \mathbb{R}$ by

$d(x, y) := ||x - y|| \ \forall \ x, y \in \mathrm{B}$

Then we call $d(x, y)$ the **distance** between x and y.

a) Show that d is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space B with distance/metric d is complete, then we call B a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 example (3), and you don't have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let B^* be the dual space of B, i.e., the space of linear functionals ℓ on B, and define a norm of ℓ by

$||\ell|| := \sup_{||x||=1} \{|\ell(x)|\}$

c) Show that the such defined norm on B^* is a norm in the sense of the norm defined on B above.

(In case you're wondering: B^* is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)

1.4.8. **Hilbert space**

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of the definition in Problem 1.4.7.

hint: Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings ℓ defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)

1.4.5.) Positive definiteness and symmetry are obvious.

Prove the triangle inequality:

By definition of $|x|$, we have $xy \le |x|\cdot|y| \ \forall x,y \in \mathbb{R}$

$\leadsto \quad 0 \le 2(x-y)(z-y) + 2|x-y|\cdot|z-y|$

$\leadsto \quad (x-z)^2 = x^2-2xz+z^2 \le x^2-2xz+z^2+2(x-y)(z-y)+2|x-y|\cdot|z-y|$

$$= x^2-2xz+z^2+2(x-y)z-2(x-y)y+2|x-y|\cdot|z-y|$$

$$= x^2-2xz+z^2+2xz-2xy+2y^2-2yz+2|x-y|\cdot|z-y|$$

$$= x^2-2xy+y^2+y^2-2yz+z^2+2|x-y|\cdot|z-y|$$

$$= (x-y)^2+(y-z)^2+2|x-y|\cdot|y-z|$$

$$= \left(|x-y|+|y-z|\right)^2$$

That $(x-z)^2 \ge 0 \leadsto$

$$|x-z| \le |x-y|+|y-z| \quad \text{triangle inequality}$$

□

1.4.6.) a) Let x_n be a sequence. Suppose $x_n \Rightarrow x^*$ and $x_n \Rightarrow y^*$.

$\rightsquigarrow \rho(x^*, y^*) \leq \rho(x^*, x_n) + \rho(y^*, x_n) \quad \forall x_n$ by the triangle ineq.

But $\lim_{n\to\infty} \rho(x^*, x_n) = \lim_{n\to\infty} \rho(y^*, x_n) = 0$

1 $\rightsquigarrow \rho(x^*, y^*) = 0 \quad \rightsquigarrow \quad x^* = y^*$ $\square$

b) Let x_n have a limit x^*: $x_n \Rightarrow x^*$

1 $\rightsquigarrow \rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*)$

Let $\delta > 0$. Then $\exists N \in \mathbb{N}: \rho(x_n, x^*) < \delta \quad \forall n > N$

Now let $\varepsilon > 0$ and $\delta = \varepsilon/2$. Then $\exists N > 0$:

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

1 provided $n, m > N$. $\square$

1.4.7) a) $d(x-y) = \|x-y\| \geq 0 \quad \forall x,y \in B$ by property (i) of $\|\ldots\|$

and $d(x-y) = 0$ iff $x-y = 0 \Leftrightarrow x = y$

⇝ positive definiteness ✓

$d(y-x) = \|y-x\| = \|-(x-y)\| = \|x-y\|$ by property (iii)
$= d(x-y)$

⇝ symmetry ✓

$d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\|$ by property (ii)
$= d(x,y) + d(y,z)$

1 ⇝ triangle inequality ✓

b) Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space and define

$$\|x\| := |x| \quad \forall x \in \mathbb{R}$$

Then $\|\ldots\| : \mathbb{R} \to \mathbb{R}$ has all of the properties required of a norm. Furthermore, §4.5 ex (3) ⇝ any Cauchy sequence has a limit ⇝ $\mathbb{R}$ is complete and hence a B-space.

Same for $\mathbb{C}$ with a norm defined by

$$\|z\| := |z| = \sqrt{z_1^2 + z_2^2}$$

1 This makes $\mathbb{C}$ a B-space (assuming completeness)

c) Check the requirements from the definition of a norm:

(i) $\|l\| = \sup\limits_{\|x\|=1} \{|l(x)|\} \Rightarrow \|l\| \geq 0$ since $|l(x)| \geq 0$

1/2

The null vector in $\mathfrak{B}^*$ is the null functional l_0 defined by $l_0(x) = 0 \ \forall x \in \mathfrak{B}$.

$\Rightarrow \|l_0\| = 0$

Conversely, let $\|l\| = 0$. Since the set of $x \in \mathfrak{B}$ with $\|x\| = 1$ spans $\mathfrak{B}$, l must equal l_0

1 $\Rightarrow \|l\| = 0$ iff $l = l_0$

(ii) $\|l_1 + l_2\| = \sup\limits_{\|x\|=1} \{|l_1(x) + l_2(x)|\} \leq \sup\{|l_1(x)| + |l_2(x)|\}$

$= \|l_1\| + \|l_2\|$

1 That is, $\mathfrak{B}^*$ inherits the triangle inequality from $\mathbb{C}$.

(iii) $\|a l\| = \sup\limits_{\|x\|=1} \{|a l(x)|\} = \sup\{|a| \cdot |l(x)|\} = |a| \cdot \sup\{l(x)\}$

1/2 $= |a| \cdot \|l\| \quad \forall a \in \mathbb{C}, l \in \mathfrak{B}^*$

$\square$

1.4.8.) a) (i) $(x,x) = (x,x)^* \in \mathbb{R}$ and $(x,x) \geq 0 \rightsquigarrow \|x\| = (x,x)^{1/2} \geq 0 \;\forall x \in H$

1/2 and $(x,x) = 0$ iff $x = \not{0} \rightsquigarrow \|x\| = 0$ iff $x = \not{0}$

(ii) $\|x+y\|^2 = (x+y, x+y) = (x,x) + (y,y) + (x,y) + (y,x)$

$= (x,x) + (y,y) + (x,y) + (x,y)^*$

$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(x,y)$

But $(\mathrm{Re}\,(x,y))^2 + (\mathrm{Im}\,(x,y))^2 = |(x,y)|^2$.

$\rightsquigarrow \mathrm{Re}\,(x,y)^2 = |(x,y)|^2 - \underbrace{(\mathrm{Im}\,(x,y))^2}_{\geq 0} \leq |(x,y)|^2$

$\overset{\text{CS (§4.7 lemma)}}{\leq} (x,x)(y,y) = \|x\|^2 \cdot \|y\|^2$

$\rightsquigarrow \|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2$

1 $\rightsquigarrow \|x+y\| \leq \|x\| + \|y\|$

(iii) $\|ax\|^2 = (ax, ax) = aa^*(x,x) = |a|^2 \cdot \|x\|^2$

1/2 $\rightsquigarrow \|ax\| = |a| \cdot \|x\| \quad \forall a \in \mathbb{C},\, x \in H \quad \rightsquigarrow \|\ldots\|$ is a norm $\square$

b) The definition $l(x) := (y,x)$ implies

(i) $l(x+z) = (y, x+z) = (y,x) + (y,z) = l(x) + l(z) \quad \forall x, z \in H$

(ii) $l(\lambda x) = (y, \lambda x) = \lambda (y,x) = \lambda\, l(x) \quad \forall x \in H,\, \lambda \in \mathbb{C}$

1 $\rightsquigarrow l$ is a linear form