Problem Assignment # 6 11/05/2020
due 11/12/2020

1.4.9. **Lorentz transformations in M_2**

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

1.4.10. **Time-like and space-like intervals**

Consider two points (ct_x, x^1, x^2, x^3) and (ct_y, y^1, y^2, y^3) in Minkowski space. The interval between the two points is called *time-like* if

$$c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad ,$$

and *space-like* if

$$c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad .$$

Show that in interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)

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1.4.11. **Special Lorentz transformations in** M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu$ (rotations)

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu$ (Lorentz boost along the x-direction)

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu$ (parity)

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu$ (time reversal)

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

1.4.12. **General Lorentz transformations in** M_4

Let D be a general Lorentz transformation in M_4.

a) Show that $|D^0{}_0| \geq 1$, and that $(D^0{}_1)^2 + (D^0{}_2)^2 + (D^0{}_3)^2 = (D^1{}_0)^2 + (D^2{}_0)^2 + (D^3{}_0)^2$.

b) Let $L_{++} = \{D \in L; \det D > 0, D^0{}_0 > 0\}$. (This is called the set of proper orthochronous Lorentz transformations, and one can show that it is a subgroup of L.) Show that any Lorentz transformation can be written as an element of L_{++} followed by either P, or T, or PT. It thus suffices to study L_{++}.

c) Show that any element of L_{++} can be written as a spatial rotation $R(\Phi, \Theta, \Psi)$ followed by a Lorentz boost $B(\alpha)$ followed by a rotation about the 3-axes followed by a rotation about the 2-axis. In a symbolic notation:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & R_2(\phi)R_3(\theta) \end{pmatrix} B(\alpha) \begin{pmatrix} 1 & 0 \\ 0 & R(\Phi, \Theta, \Psi) \end{pmatrix}$$

L_{++} is thus characterized by six parameters: 3 Euler angles Φ, Θ, Ψ, the boost parameter α, and two additional rotation angles ϕ and θ.

(7 points)

1.4.9, a) Let $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For Λ to be a Lorentz transfo, we must have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2-b^2 & ac-bd \\ ac-bd & c^2-d^2 \end{pmatrix}$$

From this obtain three constraints on the four numbers a,b,c,d:

(i) $a^2-b^2=1$

(ii) $c^2-d^2=-1$

1 (iii) $ac-bd=0$

Now consider $\sinh\phi$, which maps $\mathbb{R}$ one-to-one onto itself.

$\rightsquigarrow \forall\, b \in \mathbb{R}\ \exists!\ \phi \in \mathbb{R}: \quad \underline{b = \sinh\phi}$

(i) $\rightsquigarrow a^2 = 1+b^2 = 1+\sinh^2\phi = \cosh^2\phi \rightsquigarrow \underline{a = \sigma\cosh\phi}, \underline{\sigma=\pm1}$

Analogously, $\quad \underline{c = \sinh\psi}$

1 (ii) $\rightsquigarrow d^2 = 1+c^2 = 1+\sinh^2\psi = \cosh^2\psi \rightsquigarrow \underline{d = \tau\cosh\psi}, \underline{\tau=\pm1}$

Finally,

$$\begin{aligned}
\text{(iii)} \rightsquigarrow 0 &= \sigma\cosh\phi\sinh\psi - \tau\cosh\psi\sinh\phi \\
&= \cosh\phi\sinh\psi - \sigma\tau\cosh\psi\sinh\phi \\
&= \cosh(\sigma\tau\phi)\sinh\psi - \cosh\psi\sinh(\sigma\tau\phi) \\
&= \sinh(\psi - \sigma\tau\phi) \quad \rightsquigarrow \underline{\psi = \sigma\tau\phi}
\end{aligned}$$

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$$\begin{aligned}
\rightsquigarrow \underline{\Lambda_{\sigma,\tau}(\phi)} &= \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sinh(\sigma\tau\phi) & \tau\cosh(\sigma\tau\phi) \end{pmatrix} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\tau\sinh\phi & \tau\cosh\phi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \\
&= \underline{\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\Lambda(\phi)\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}} \quad \text{with} \quad \underline{\Lambda(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}, \ \phi \in \mathbb{R}}
\end{aligned}$$

1 and $\sigma,\tau = \pm1$ is the most general element of $O(1,1)$.

b) (i) $\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix}\begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$

$\rightsquigarrow$ $D(\phi_1)D(\phi_2) = D(\phi_1+\phi_2)$ closure ✓

(ii) Matrix multiplication is associative ✓

(iii) $D(\phi=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$ neutral element ✓

(iv) $D(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$

and $\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightsquigarrow$ $D(-\phi) = (D(\phi))^{-1}$ inverse ✓

$\rightsquigarrow$ $\underline{\{D(\phi)\}}$ is a $\underline{\text{group } SO^+(1,1)}$

1 (i), (iii), (iv) provide the isomorphism $\underline{SO^+(1,1) \cong \mathbb{R}(+)}$

c) $e^{J\phi} = \mathbb{1}_2 + J\phi + \frac{1}{2}J^2\phi^2 + \dots$

and $\cosh\phi = 1 + \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 + \dots$, $\sinh\phi = \phi + \frac{1}{3!}\phi^3 + \dots$

Try $\underline{J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\rightsquigarrow$ $J^2 = \mathbb{1}_2$, $J^3 = J$, etc.

$\rightsquigarrow$ $\underline{\underline{e^{J\phi}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\phi^2 & 0 \\ 0 & \frac{1}{2}\phi^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}\phi^3 \\ \frac{1}{3!}\phi^3 & 0 \end{pmatrix} + \dots$

$\underline{\underline{= \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}}}$ ✓

1 $\rightsquigarrow$ $\underline{J \text{ is the generator}}$ of $\underline{\underline{SO^+(1,1)}}$.

1.4.10.) Let $c^2(t_x-t_y)^2 \geq (x^1-y^1)^2+(x^2-y^2)^2+(x^3-y^3)^2$

1 $\Leftrightarrow$ $\boxed{(x^\mu-y^\mu)\,g_{\mu\nu}\,(x^\nu-y^\nu) \geq 0}$ (*)

Now consider a Lorentz trafo $x^\mu = \Lambda^\mu{}_\nu \tilde{x}^\nu$

(*) $\rightsquigarrow$ $(\tilde{x}^\alpha-\tilde{y}^\alpha)\underbrace{\Lambda^\mu{}_\alpha\, g_{\mu\nu}\, \Lambda^\nu{}_\beta}_{=g_{\alpha\beta}}(\tilde{x}^\beta-\tilde{y}^\beta) \geq 0$

$\rightsquigarrow$ $(\tilde{x}^\mu-\tilde{y}^\mu)\,g_{\mu\nu}\,(\tilde{x}^\nu-\tilde{y}^\nu) \geq 0$

1 $\rightsquigarrow$ (*) holds in all inertial frames

1.4.11 .10) i) $\underline{R^\mu{}_\alpha g_{\mu\nu} R^\nu{}_\beta} = \left(\begin{array}{c|c}1&0\\ \hline 0&R^T\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&R\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&R^T\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-R\end{array}\right)$

$$= \left(\begin{array}{c|c}1&0\\ \hline 0&-R^TR\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right) = \underline{\underline{g_{\alpha\beta}}} \quad ✓$$

ii) $\underline{\mathbb{B}^\mu{}_\alpha g_{\mu\nu} \mathbb{B}^\nu{}_\beta} = \begin{pmatrix}\cosh\alpha & 0 & 0 & \sinh\alpha\\ 0&1&0&0\\ 0&0&1&0\\ \sinh\alpha&0&0&\cosh\alpha\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$

$$\times\begin{pmatrix}\cosh\alpha & 0 & 0 & \sinh\alpha\\ 0&1&0&0\\ 0&0&1&0\\ \sinh\alpha&0&0&\cosh\alpha\end{pmatrix}$$

$$= \begin{pmatrix}\cosh^2\alpha-\sinh^2\alpha & 0 & 0 & \cosh\alpha\sinh\alpha-\sinh\alpha\cosh\alpha\\ 0&-1&0&0\\ 0&0&-1&0\\ \sinh\alpha\cosh\alpha-\cosh\alpha\sinh\alpha&0&0&\sinh^2\alpha-\cosh^2\alpha\end{pmatrix}$$

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$$= \begin{pmatrix}1&&&0\\&-1&&\\&&-1&\\0&&&-1\end{pmatrix} = \underline{\underline{g_{\alpha\beta}}} \quad ✓$$

iii) $\underline{P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\beta} = \left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right)\left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right) = \left(\begin{array}{c|c}1&0\\ \hline 0&-1\end{array}\right) = \underline{\underline{g_{\alpha\beta}}} \quad ✓$

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iv) $\underline{T^\mu{}_\alpha g_{\mu\nu} T^\nu{}_\beta} = (-)^2 P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\beta = \underline{\underline{g_{\alpha\beta}}} \quad ✓$

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b) $R^\mu{}_\nu$ leaves the time coordinate invariant, and we know that the $R^i{}_j$ form a group (see PHYS 611) $\rightarrow$ $\{R^i{}_j\}$ is a subgroup of L. The Lorentz boosts $\mathbb{B}^\mu{}_\nu$ are a subgroup of L according to Problem 1.8.

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c) group tables:

	$\mathbb{1}$	P
$\mathbb{1}$	$\mathbb{1}$	P
P	P	$\mathbb{1}$

and

	$\mathbb{1}$	T
$\mathbb{1}$	$\mathbb{1}$	T
T	T	$\mathbb{1}$

$\rightarrow$ $\{\mathbb{1}, P\}$, and $\{\mathbb{1}, T\}$ are subgroups of L under matrix multiplication

	$\mathbb{1}$	P	T	PT
$\mathbb{1}$	$\mathbb{1}$	P	T	PT
P	P	$\mathbb{1}$	PT	T
T	T	PT	$\mathbb{1}$	P
PT	PT	T	P	$\mathbb{1}$

$\rightarrow$ $\{\mathbb{1}, P, T, PT\}$ is also a subgroup of L

1.4.12.) a) With
$$D = \left(\begin{array}{c|c} \delta & \vec{a} \\ \hline \vec{b} & d \end{array}\right)$$

multiply with g from left

defining property: $g = D^T g D \rightsquigarrow \underline{1 = (g D^T g) D}$ (*)

$\rightsquigarrow D^{-1} = g D^T g \rightsquigarrow \underline{1 = D (g D^T g)}$ (**)

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We have $g D^T g = \left(\begin{array}{c|c} \delta & -\vec{b} \\ \hline -\vec{a} & d^T \end{array}\right)$

(**) $\rightsquigarrow$
$\underline{1} = \delta^2 - \vec{a}^2$
$0 = \delta \vec{b} - \vec{a} d^T$
$\underline{1} = d d^T - \vec{b}^2$

(*) $\rightsquigarrow$
$\underline{1} = \delta^2 - \vec{b}^2$
$0 = \delta \vec{a} - \vec{b} d$
$\underline{1} = d^T d - \vec{a}^2$

In particular, $\delta^2 = 1 + \vec{a}^2 \geq 1 \rightsquigarrow \underline{\underline{|D^0{}_0| \geq 1}}$

1 It also follows that $|\vec{a}| = |\vec{b}|$

b) We also know $\det D = \pm 1$

$\rightsquigarrow$ We can classify L as follows:

$L_{++} = \{ D \in L ;\ \det D = 1,\ D^0{}_0 \geq 1 \}$

$L_{+-} = \{ D \in L ;\ \det D = 1,\ D^0{}_0 \leq -1 \}$

$L_{-+} = \{ D \in L ;\ \det D = -1,\ D^0{}_0 \geq 1 \}$

$L_{--} = \{ D \in L ;\ \det D = -1,\ D^0{}_0 \leq -1 \}$

Clearly, $L = L_{++} \cup L_{+-} \cup L_{-+} \cup L_{--}$

and $L_{\sigma\tau} \cap L_{\sigma'\tau'} = \emptyset$ if $(\sigma, \tau) \neq (\sigma', \tau')$

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P14.12-2

Now let $\Lambda = \begin{pmatrix} \delta & \vec{c} \\ \vec{b} & d \end{pmatrix} \in L$. Then

$$P\Lambda = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)\left(\begin{array}{c|c} \delta & \vec{c} \\ \hline \vec{b} & d \end{array}\right) = \left(\begin{array}{c|c} \delta & \vec{c} \\ \hline -\vec{b} & -d \end{array}\right)$$

changes the sign of $\det \Lambda$, leaves $\Lambda^0{}_0$ invariant, and provides an isomorphism to L_{-+}

Similarly,

$$T\Lambda = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} \delta & \vec{c} \\ \hline \vec{b} & d \end{array}\right) = \left(\begin{array}{c|c} -\delta & -\vec{c} \\ \hline \vec{b} & d \end{array}\right)$$

changes the sign of both $\det \Lambda$ and $\Lambda^0{}_0$

$$PT\Lambda = -\Lambda = \left(\begin{array}{c|c} -\delta & -\vec{c} \\ \hline -\vec{b} & -d \end{array}\right)$$

changes the sign of $\Lambda^0{}_0$ but leaves $\det \Lambda$ invariant

$\rightsquigarrow$ $\underline{L_{-+} = PL_{++}, \quad L_{--} = TL_{++}, \quad L_{+-} = PTL_{++}}$

$\rightsquigarrow$ It suffices to study L_{++}

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c) Consider the 3-vector $\vec{b} = (b^1, b^2, b^3)$.
We know from the theory of $SO(3)$ (see, e.g., PHYS 612 § 3.2)
that there exist two rotations, $R_3(\vartheta)$ about the 3-axis, and
$R_2(\varphi)$ about the 2-axis, such that

$$R_3(\vartheta) R_2(\varphi)\, \vec{b} = (\tilde{b}, 0, 0) \quad \text{with} \quad \tilde{b}^2 = (b^1)^2 + (b^2)^2 + (b^3)^2$$

$$\rightsquigarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_3(\vartheta) R_2(\varphi) \end{array}\right) \Lambda = \left(\begin{array}{c|ccc} \delta & \cdot & \cdot & \cdot \\ \hline \tilde{b} & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{array}\right) =: \tilde{\Lambda}$$

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where $\delta^2 = 1 + \tilde{b}^2$ $\rightsquigarrow$ $\exists\, \alpha \in \mathbb{R}$: $\delta = \cosh \alpha$
$\tilde{b} = \sinh \alpha$

→ There exists a Lorentz boost such that

$$\Lambda(\alpha)\,\hat{D} = \left(\begin{array}{c|c} 1 & \tilde{c} \\ \hline 0 & \\ 0 & \hat{d} \\ 0 & \end{array}\right)$$

But we know from part c) that $|\tilde{c}| = 0$ since the two 3-vectors must have the same length.

→ $$\Lambda(\alpha)\,R_3(\vartheta)\,R_2(\varphi)\,D = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{d} \end{array}\right)$$

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with $\hat{d}$ a Euclidean orthogonal transfo. But the latter we know, again from the theory of $SO(3)$ (e.g., PH45(1?)), can be represented by rotations through three Euler angles.

→ A most general rotation $R(\Phi, \Theta, \Psi)$, followed by a general Lorentz boost along the 1-axis, $\Lambda(\alpha)$, followed by rotations $R_3(\vartheta)$ and $R_2(\varphi)$, represent all of L_{++}. That is, if $D \in L_{++}$, then there exist Euler angles Φ, Θ, Ψ, a Lorentz boost parameter α, and rotation angles ϑ and φ such that

$$D = R_2(\varphi)\,R_3(\vartheta)\,\Lambda(\alpha)\,R(\Phi, \Theta, \Psi)$$

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