Problem Assignments # 7

11/12/2020
due 11/19/2020

1.5.1. **Transformations of tensor fields**

a) Consider a covariant rank-n tensor field $t_{i_1 \ldots i_n}(x)$ and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how $\tilde{t}_{i_1 \ldots i_n}(\tilde{x})$ is related to $t_{i_1 \ldots i_n}(x)$.

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector x_i (the case $n = 1$), and (ii) the covariant components of the metric tensor g_{ij}.

(4 points)

1.5.2. **Curl and divergence**

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

1.5.3. **Tensor products, and tensor traces**

Prove Propositions 1 and 2 from ch. 1 §5.3.

(4 points)

2.2.1. **Lindhard function**

Consider the function $f : \mathbb{C} \to \mathbb{C}$ (which plays an important role in the theory of many-electron systems) defined by

$$f(z) = \log\left(\frac{z-1}{z+1}\right)$$

The *spectrum* $f'' : \mathbb{R} \to \mathbb{R}$ and the *reactive part* $f' : \mathbb{R} \to \mathbb{R}$ of f are defined by

$$f''(\omega) := \frac{1}{2i}\left[f(\omega + i0) - f(\omega - i0)\right] \qquad , \qquad f'(\omega) := \frac{1}{2}\left[f(\omega + i0) + f(\omega - i0)\right]$$

a) Show that f' and f'' are indeed real-valued functions.

b) Determine f'' and f' explicitly, and plot them for $-3 < \omega < 3$.

c) Show that

$$\int_{\infty}^{\infty} \frac{d\omega}{\pi}\, \frac{f''(\omega)}{\omega - z} = f(z)$$

(5 points)

1.5.1.) a) There are various ways to do this. One option is to start with the transformation property of contravariant tensor fields, §5.1, and use the metric tensor to lower the indices:

$$\underline{\tilde t_{i_1 \dots i_N}(\tilde x)} = \tilde g_{i_1 j_1} \cdots \tilde g_{i_N j_N} \tilde t^{j_1 \dots j_N}(\tilde x)$$

$$\overset{§5.1(x) + \tilde g = g}{=} g_{i_1 j_1} \cdots g_{i_N j_N} D^{j_1}{}_{k_1} \cdots D^{j_N}{}_{k_N} t^{k_1 \dots k_N}(x)$$

$$= g_{i_1 j_1} \cdots g_{i_N j_N} D^{j_1}{}_{k_1} \cdots D^{j_N}{}_{k_N} g^{k_1 l_1} \cdots g^{k_N l_N} t_{l_1 \dots l_N}(x)$$

$$= (gD)_{i_1 k_1} \cdots (gD)_{i_N k_N} g^{k_1 l_1} \cdots g^{k_N l_N} t_{l_1 \dots l_N}(x)$$

$$\overset{gD = (D^T)^{-1} g}{=} ((D^T)^{-1} g)_{i_1 k_1} \cdots ((D^T)^{-1} g)_{i_N k_N} g^{k_1 l_1} \cdots g^{k_N l_N} t_{l_1 \dots l_N}(x)$$

$$= ((D^T)^{-1})_{i_1}{}^{m_1} g_{m_1 k_1} \cdots ((D^T)^{-1})_{i_N}{}^{m_N} g_{m_N k_N} g^{k_1 l_1} \cdots g^{k_N l_N} t_{l_1 \dots l_N}(x)$$

$$= ((D^T)^{-1})_{i_1}{}^{m_1} \underbrace{g_{m_1 k_1} g^{k_1 l_1}}_{= \delta_{m_1}{}^{l_1}} \cdots ((D^T)^{-1})_{i_N}{}^{m_N} \underbrace{g_{m_N k_N} g^{k_N l_N}}_{= \delta_{m_N}{}^{l_N}} t_{l_1 \dots l_N}(x)$$

(2) $$\underline{= ((D^T)^{-1})_{i_1}{}^{j_1} \cdots ((D^T)^{-1})_{i_N}{}^{j_N} t_{j_1 \dots j_N}(x)}$$

This is §5.1 (x) with $D^{i_n}{}_{j_n}$ replaced by $((D^T)^{-1})_{i_n}{}^{j_n}$ $(n = 1, \dots, N)$

b) Special case $N = 1$:

(1) $$\underline{\tilde x_i} = \tilde g_{ij} \tilde x^j \overset{§4.8.2}{=} \tilde g_{ij} D^j{}_l x^l \overset{\tilde g = g}{=} (gD)_{il} x^l \overset{gD = D^{T-1} g}{=} ((D^T)^{-1} g)_{il} x^l$$

$$= ((D^T)^{-1})_i{}^j g_{jl} x^l = \underline{((D^T)^{-1})_i{}^j x_j} \quad ✓$$

(1) $$\tilde g_{ij} = g_{ik} g_{jl} \tilde g^{kl} \overset{§5.1}{=} g_{ik} g_{jl} D^k{}_m D^l{}_n g^{mn} = (gD)_{im} (gD)_{jn} g^{mo} g^{np} g_{op}$$

$$= (D^{T-1} g)_{im} g^{mo} (D^{T-1} g)_{jn} g^{np} g_{op} = ((D^T)^{-1})_i{}^o ((D^T)^{-1})_j{}^p g_{op} \quad ✓$$

1.5.2.) Consider the curl as defined in § 5.2:

$$c^i(x) = \varepsilon^{ijl}\,\partial_j v_l(x)$$

$$\rightsquigarrow \underline{\tilde{c}^i(\tilde{x})} = \tilde{\varepsilon}^{ijl}\,\tilde{\partial}_j \tilde{v}_l(\tilde{x})$$

Prob 2.5

$$\overset{\downarrow}{=} \tilde{\varepsilon}^{ijl}(D^{-1})^e{}_j\,\partial_e (D^{-1})^m{}_l\, v_m(x)$$

$$= \delta^i{}_n\, \tilde{\varepsilon}^{njl}(D^{-1})^e{}_j (D^{-1})^m{}_l\, \partial_e v_m(x)$$

$$= D^i{}_p (D^{-1})^p{}_n\, \tilde{\varepsilon}^{njl}(D^{-1})^e{}_j (D^{-1})^m{}_l\, \partial_e v_m(x)$$

$$= D^i{}_p\, \underbrace{(D^{-1})^p{}_n (D^{-1})^e{}_j (D^{-1})^m{}_l\, \tilde{\varepsilon}^{njl}}_{=(\det D)\,\varepsilon^{pem} \text{ by § 5.1 remark (a)}}\, \partial_e v_m(x)$$

$$= (\det D)\, D^i{}_p\, \underbrace{\varepsilon^{pem}\,\partial_e v_m(x)}_{=c^p(x)}$$

(2) $$\underline{= (\det D)\, D^i{}_p\, c^p(x)}$$

$\rightsquigarrow$ $c^i(x)$ transforms as a pseudovector field.

Now the divergence: $d(x) = \partial_i v^i(x)$

$$\rightsquigarrow \underline{\tilde{d}(\tilde{x})} = \tilde{\partial}_i \tilde{v}^i(\tilde{x}) = ((D^T)^{-1})_i{}^j\,\partial_j\, D^i{}_l\, v^l(x)$$

$$= \underbrace{(D^T)_l{}^i ((D^T)^{-1})_i{}^j}_{=\delta_l{}^j}\,\partial_j v^l(x)$$

(1) $$\underline{= \partial_l v^l(x)}$$ $\rightsquigarrow$ $d(x)$ transforms as a scalar field

1.5.2) Consider the tensor product

$$u^{i_1\dots i_{N+M}} = s^{i_1\dots i_N}\, t^{i_{N+1}\dots i_{N+M}}$$

Let both s and t be tensors. Then

$$\underline{\tilde u^{i_1\dots i_{N+M}}} = \tilde s^{i_1\dots i_N}\, \tilde t^{i_{N+1}\dots i_{N+M}}$$

$$= D^{i_1}{}_{j_1}\cdots D^{i_N}{}_{j_N}\, s^{j_1\dots j_N}\, D^{i_{N+1}}{}_{j_{N+1}}\cdots D^{i_{N+M}}{}_{j_{N+M}}\, t^{j_{N+1}\dots j_{N+M}}$$

$$= \underline{D^{i_1}{}_{j_1}\cdots D^{i_{N+M}}{}_{j_{N+M}}\, u^{j_1\dots j_{N+M}}} \qquad (*)$$

→ u is a rank-(N+M) tensor

Now suppose s is a pseudotensor and t a tensor, or vice versa. Then the rhs of (*) gets multiplied

1 by det D → u is a pseudotensor

If both s and t are pseudotensors, then the rhs of (*)

1 gets multiplied by $(\det D)^2 = 1$ → u is a tensor

This proves prop. 1 from §5.2.

Now consider

$$u^{k_1\dots k_N} = t_i{}^{i k_1\dots k_N} = g_{ij}\, t^{ij k_1\dots k_N}$$

Let t be a tensor. Then

$$\underline{\tilde u^{k_1\dots k_N}} = \tilde g_{ij}\, \tilde t^{ij k_1\dots k_N}$$

1 $$\overset{\text{Problem 2.5}}{=} ((D^{-1})^T)_i{}^o\, ((D^{-1})^T)_j{}^p\, g_{op}\, D^i{}_m D^j{}_n D^{k_1}{}_{l_1}\cdots D^{k_N}{}_{l_N}\, t^{mn l_1\dots l_N}$$

$$= \underbrace{(D^{-1})^o{}_i D^i{}_m}_{=\delta^o{}_m}\, \underbrace{(D^{-1})^p{}_j D^j{}_n}_{=\delta^p{}_n}\, D^{k_1}{}_{l_1}\cdots D^{k_N}{}_{l_N}\, g_{op}\, t^{mn l_1\dots l_N}$$

$$= D^{k_1}{}_{l_1} \cdots D^{k_N}{}_{l_N} \, g_{mn} \, t^{mn l_1 \ldots l_N}$$

$$= D^{k_1}{}_{l_1} \cdots D^{k_N}{}_{l_N} \, u^{l_1 \ldots l_N} \qquad (**)$$

$\Rightarrow$ u is a tensor of rank N

If t is a pseudotensor, then the rhs of (**) gets multiplied by det D

$\Rightarrow$ u is a pseudotensor of rank N

(1) This proves prop 2 from § 5.3

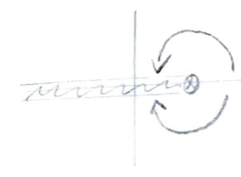
2.2.1) a) $f(\omega - i0) = f(\omega + i0)^*$, und $z + z^* = z' + iz'' + z' - iz'' = 2z' \in \mathbb{R}$

und $\frac{1}{i}(z - z^*) = 2z'' \in \mathbb{R}$

1 $\rightsquigarrow f'(\omega), f''(\omega) \in \mathbb{R}$

b) $f(z) = \ln \frac{z-1}{z+1} = \ln(z-1) - \ln(z+1)$

$\rightsquigarrow f''(\omega) = \frac{1}{2i}\left[\ln(\omega - 1 + i0) - \ln(\omega - 1 - i0) - \ln(\omega + 1 + i0) + \ln(\omega + 1 - i0)\right]$



$$\rightsquigarrow \underline{2i f''(\omega)} = \Theta(\omega < 1)\left[i\pi - (-i\pi)\right] - \Theta(\omega < -1)\left[i\pi - (-i\pi)\right]$$

$$= \left(\Theta(\omega < 1) - \Theta(\omega < -1)\right) 2i\pi$$

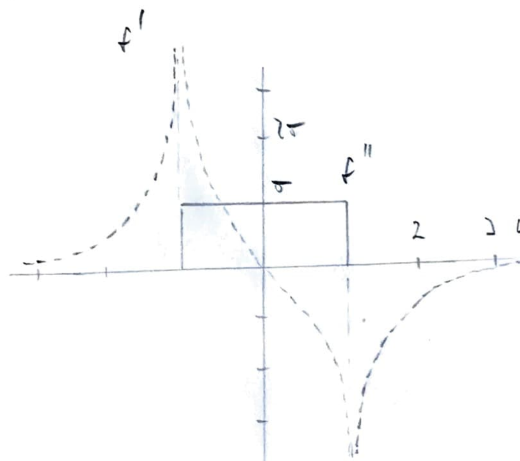
$$= \underline{\Theta(\omega^2 < 1)\, 2i\pi}$$

1 $\rightsquigarrow \underline{\underline{f''(\omega) = \pi\, \Theta(\omega^2 < 1)}}$

$$f'(\omega) = \frac{1}{2}\left[\ln(\omega - 1 + i0) + \ln(\omega - 1 - i0) - \ln(\omega + 1 + i0) - \ln(\omega + 1 - i0)\right]$$

$$\rightsquigarrow \underline{2f'(\omega)} = \underline{2\ln|\omega - 1| - 2\ln|\omega + 1|}$$

1 $\rightsquigarrow \underline{\underline{f'(\omega) = \ln\left|\frac{\omega - 1}{\omega + 1}\right|}}$



c) $\underline{\underline{\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{f''(\omega)}{\omega - z}}} =$

$$= \int_{-1}^{1} d\omega \frac{1}{\omega - z} = \ln(\omega - z)\Big|_{-1}^{1}$$

$$= \ln(1 - z) - \ln(-1 - z) = \ln \frac{1 - z}{-1 - z}$$

1 $= \ln \frac{z-1}{z+1} = \underline{\underline{f(z)}}$