Problem Assignments # 8 11/19/2020
due 11/26/2020

2.2.2. **Another causal function**

The function considered in Problem 2.2.1 is an example of a class of complex functions called *causal functions* that are important for the theory of many-particle systems. Another member of this class is

$$g(z) = \sqrt{z^2 - 1} - z$$

Determine the spectrum and the reactive part of $g(z)$, and plot them for $-3 < \omega < 3$.

(3 points)

2.2.3. **Proof of the Cauchy-Riemann Theorem**

Prove the Cauchy-Riemann theorem from ch.2 §2.2:

a) Let $f(z) = f'(z', z'') + if''(z', z'')$ be analytic everywhere in $\Omega \subseteq \mathbb{C}$. Show that the Cauchy-Riemann equations

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \qquad \text{and} \qquad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}$$

hold $\forall z \in \Omega$.

hint: Start with the difference quotient $(f(z) - f(z_0))/(z - z_0)$ and require that it's limit for $z \to z_0$ exists if z_0 is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

b) Let the Cauchy-Riemann equations hold in a point $z_0 \in \Omega$. Show that this implies that f is analytic in the point z_0.

hint: Consider $f(z) - f(z_0)$ and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series about z_0.

(8 points)

2.2.4. **Exponentials**

Consider the exponential function

$$f(z) = e^z = e^{z' + iz''}$$

a) Show that $f(z)$ is analytic everywhere in $\mathbb{C}$.

b) Convince your self explicitly that the real and imaginary parts of f obey Laplace's differential equation.

c) Show that $df/dz|_z = f(z)$.

d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad , \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

are analytic everywhere in $\mathbb{C}$, and that

$$\frac{d}{dz}\cos z = -\sin z \quad , \quad \frac{d}{dz}\sin z = \cos z \ .$$

(4 points)

. . . **/over**

2.3.1. **Laurent series**

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter the theorem in ch. 2 §3.2.

(3 points)

2.2.2.)

$$f(z) = \sqrt{z^2-1} - z = \sqrt{z+1}\,\sqrt{z-1} - z$$

$$f''(\omega) = \frac{1}{2i}\left[\sqrt{\omega+1+i0}\,\sqrt{\omega-1+i0} - \sqrt{\omega+1-i0}\,\sqrt{\omega-1-i0}\right] - \frac{1}{2i}(\omega+i0-\omega+i0)$$

$$= \frac{1}{2i}\Big[\big(\Theta(\omega>-1)(\omega+1)^{1/2} + \Theta(\omega<-1)|\omega+1|^{1/2} i\big)\big(\Theta(\omega>1)(\omega-1)^{1/2} + \Theta(\omega<1)|\omega-1|^{1/2} i\big) - cc\Big]$$

$$= \Theta(\omega>-1)\,\Theta(\omega<1)\,|\omega^2-1|^{1/2} + \underbrace{\Theta(\omega<-1)\,\Theta(\omega>1)}_{=0}\,|\omega^2-1|^{1/2}$$

1

$$= \Theta(\omega^2<1)\,(1-\omega^2)^{1/2}$$

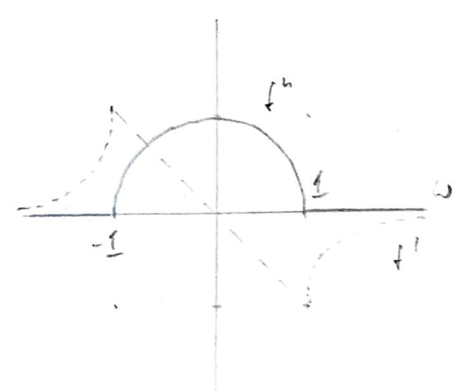
$$f'(\omega) = \frac{1}{2}\left[f(\omega+i0) + f(\omega-i0)\right] = \mathrm{Re}\, f(\omega+i0)$$

$$= \Theta(\omega>-1)\,\Theta(\omega>1)\sqrt{\omega^2-1} - \Theta(\omega<-1)\,\Theta(\omega<1)\sqrt{\omega^2-1} - \omega$$

1

$$= \Theta(\omega^2<1)(-\omega) + \Theta(\omega^2>1)\left(\sqrt{\omega^2-1}\,\mathrm{sgn}\,\omega - \omega\right)$$

1



2.2.3 a) Consider the difference quotient

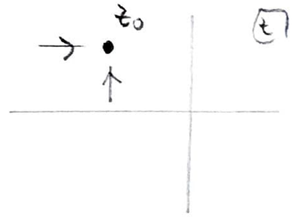
$$\frac{\Delta f}{\Delta z} = \frac{f^1(z^1,z^2) + i f^2(z^1,z^2) - f^1(z_0^1,z_0^2) - i f^2(z_0^1,z_0^2)}{z^1 + i z^2 - z_0^1 - i z_0^2}$$

for f in the vicinity of a point $z_0 \in R$.

For the limit $\frac{df}{dz} = \lim_{z \to z_0} \frac{\Delta f}{\Delta z}$ to exist, it must exist for f^1 and f^2 separately, and it must exist for $z \to z_0$ along any path. In particular, it must exist for $z \to z_0$ along the real axis and along the imaginary axis.

1 ~> If f is differentiable in z_0, then the derivatives $\partial f^1/\partial z^1$, $\partial f^1/\partial z^2$, $\partial f^2/\partial z^1$, $\partial f^2/\partial z^2$ all must exist in z_0

Now approach z_0 parallel to the real axis, i.e., for fixed z^2:



1
$$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z^2=\text{const}}} = \left.\frac{\partial f^1}{\partial z^1}\right|_{z_0} + i \left.\frac{\partial f^2}{\partial z^1}\right|_{z_0} \quad (*)$$

Now approach for fixed z^1:

1
$$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z^1=\text{const}}} = \left.\frac{\partial f^1}{i\,\partial z^2}\right|_{z_0} + i \left.\frac{\partial f^2}{i\,\partial z^2}\right|_{z_0} = \left.\frac{\partial f^2}{\partial z^2}\right|_{z_0} - i \left.\frac{\partial f^1}{\partial z^2}\right|_{z_0} \quad (**)$$

That if f is differentiable in z_0, then $(*) = (**)$

1 ~> $\boxed{\frac{\partial f^1}{\partial z^1} = \frac{\partial f^2}{\partial z^2}}$ and $\boxed{\frac{\partial f^2}{\partial z^1} = -\frac{\partial f^1}{\partial z^2}}$ ~> The Cauchy-Riemann eqs are necessary

b) Consider the numerator of the difference quotient,

$$f(z) - f(z_0) = f'(z', z'') + i f''(z', z'') - f'(z_0', z_0'') - i f''(z_0', z_0'')$$

and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series:

$$f'(z', z'') - f'(z_0', z_0'') = \frac{\partial f'}{\partial z'}\Big|_{z_0} (z' - z_0') + \frac{\partial f'}{\partial z''}\Big|_{z_0} (z'' - z_0'') + \dots$$

$$f''(z', z'') - f''(z_0', z_0'') = \frac{\partial f''}{\partial z'}\Big|_{z_0} (z' - z_0') + \frac{\partial f''}{\partial z''}\Big|_{z_0} (z'' - z_0'') + \dots$$

$$\rightsquigarrow \underline{\frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{z - z_0}\left[\frac{\partial f'}{\partial z'}\Big|_{z_0} (z' - z_0') + \frac{\partial f'}{\partial z''}\Big|_{z_0} (z'' - z_0'') + i \frac{\partial f''}{\partial z'}\Big|_{z_0} (z' - z_0') + i \frac{\partial f''}{\partial z''}\Big|_{z_0} (z'' - z_0'')\right]$$

1

+ (terms that vanish as $z \to z_0$)

$$\stackrel{\text{CR-eqs}}{=} \frac{1}{z - z_0}\left[\frac{\partial f'}{\partial z'}\Big|_{z_0} (z' - z_0' + i z'' - i z_0'') + \frac{\partial f'}{\partial z''} (z'' - z_0'' - i z' + i z_0')\right]$$

$$= \frac{\partial f'}{\partial z'}\Big|_{z_0} \frac{z - z_0}{z - z_0} - i \frac{\partial f'}{\partial z''}\Big|_{z_0} \frac{z - z_0}{z - z_0}$$

1

$$= \left(\frac{\partial f'}{\partial z'} - i \frac{\partial f'}{\partial z''}\right)\Big|_{z_0} \stackrel{\text{CR-eqs}}{=} \underline{\frac{\partial f'}{\partial z'} + i \frac{\partial f''}{\partial z'}}$$ + (terms that vanish as $z \to z_0$)

1

But our premise was that the CR-eqs hold

$\rightsquigarrow$ the rhs exists

$\rightsquigarrow$ the limit $\lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{df}{dz}\Big|_{z_0}$ exists and is independent of how z_0 is approached.

1 $\rightsquigarrow$ <u>The Cauchy-Riemann eqs are sufficient</u>

2.2.4.) a) $f(z) = e^z = e^{z'+iz''} = e^{z'}e^{iz''} = e^{z'}(\cos z'' + i\sin z'')$

$\rightarrow$ $\underline{f'(z',z'') = e^{z'}\cos z''}$

$\underline{f''(z',z'') = e^{z'}\sin z''}$

$\rightarrow$ $\underline{\frac{\partial f'}{\partial z'}} = e^{z'}\cos z'' = \underline{\frac{\partial f''}{\partial z''}}$

$\underline{\frac{\partial f'}{\partial z''}} = -e^{z'}\sin z'' = \underline{-\frac{\partial f''}{\partial z'}}$

1 $\rightarrow$ f is analytic on $\mathbb{C}$

b) $\underline{\frac{\partial^2 f'}{\partial z'^2} + \frac{\partial^2 f'}{\partial z''^2}} = e^{z'}\cos z'' - e^{z'}\cos z'' = \underline{0}$

1 $\underline{\frac{\partial^2 f''}{\partial z'^2} + \frac{\partial^2 f''}{\partial z''^2}} = e^{z'}\sin z' - e^{z'}\sin z' = \underline{0}$

c) Approach z along the real axis

$\rightarrow$ $\underline{\frac{df}{dz}(z)} = \frac{\partial f'}{\partial z'}\Big|_z + i\frac{\partial f''}{\partial z'}\Big|_z = e^{z'}\cos z'' + ie^{z'}\sin z'' = e^{z'}(\cos z'' + i\sin z'')$

1 $= e^{z'}e^{iz''} = e^z = \underline{f(z)}$ $\square$

d) Consider $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

e^z is analytic everywhere $\rightarrow$ so is $e^{\pm iz}$ $\rightarrow$ so are $\cos z$ and $\sin z$

$\frac{d}{dz}\cos z = \frac{i}{2}(e^{iz} - e^{-iz}) = \frac{-1}{2i}(e^{iz} - e^{-iz}) = \underline{-\sin z}$

$\frac{d}{dz}\sin z = \frac{i}{2i}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = \underline{\cos z}$

2.3.1.)

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\,\frac{1}{2i+(z-i)}$$

1

$$= \frac{1}{z-i}\,\frac{1}{2i}\,\frac{1}{1+\frac{1}{2i}(z-i)}$$

Now [illegible] and

$$\frac{1}{1+\frac{1}{2i}(z-i)} = \sum_{n=0}^{\infty} (-)^n \frac{(z-i)^n}{(2i)^n}$$

$$\Rightarrow f(z) = \frac{1}{z-i}\,\frac{1}{2i}\sum_{n=0}^{\infty} (-)^n \frac{(z-i)^n}{(2i)^n} = \sum_{n=0}^{\infty} (-)^n \frac{(z-i)^{n-1}}{(2i)^{n+1}}$$

1

$$= \sum_{n=-1}^{\infty} \frac{(-)^{n+1}}{(2i)^{n+2}} (z-i)^n$$

$\Rightarrow$ The coefficients f_n in the Laurent series are

1

$$f_n = \begin{cases} \dfrac{(-)^{n+1}}{(2i)^{n+2}} & \text{for } n \geq -1 \\ 0 & \text{for } n < -1 \end{cases}$$