PHYS 622 Classical Electrodynamics W 2021

Problem Assignment # 5 / Midterm 01/29/2021
due 02/12/2021

This assignment is in lieu of Assignment # 5. It doubles as the Midterm and overlaps with Assignment # 4, so you have two weeks to work on it. You can use any inanimate resource you like, but please don't consult any live resources. If you have questions, send me email. I will either answer publicly, or I will privately tell you that I won't answer the question. Submit no later than 4pm on Friday, 2/12/2021 by email to dbelitz@uoregon.edu, NOT to your grader.

0.2.9 Relativistic Kepler problem

Consider the motion of a point mass m in an attractive Coulomb potential centered at the origin within the framework of special relativity:

$$L = m\,c^2 - m\,c^2\sqrt{1-(\vec{v}/c)^2} + \alpha/r \quad , \quad (\alpha > 0\ ,\ r = |\vec{x}|) \quad .$$

Let the z-component of the angular momentum be $\ell \geq 0$.

a) Use the conservation laws for the energy and the angular momentum to find the radial equation of motion,

$$\dot{r}^2 = c^2 f(r; E, \ell)$$

and the equation for the sectorial velocity

$$r^2\,\dot{\phi} = g(r; E, \ell)$$

where ϕ is the azimuthal angle. Explicitly determined the functions f and g and show that in the nonrelativistic limit they correctly reduce to the Galilean case.

hint: Take the angular momentum $\vec{\ell}$ to point in the z-direction. You know from Mechanics that the rotational invariance of L implies that the orbit lies in a plane perpendicular to $\vec{\ell}$; you can use that without proof. Use polar coordinates in the orbital plane.

b) Assume $\ell = 0$, and let $\dot{r}(t=0) = 0$, $r(t=0) = 2a$. Discuss and draw $\dot{r}$ as a function of r, and compare with the Galilean case. Determine the oscillation period $T(a)$ in terms of a dimensionless integral that depends only on the parameter $\xi = \alpha/2amc^2$. Discuss your result in the nonrelativistic and ultrarelativistic limits ($\xi \to 0$ and $\xi \to \infty$, respectively). Show that Kepler's third law gets modified within special relativity, and plot $T/T_{\xi=0}$ as a function of ξ.

c) Now assume $\ell > 0$. Use $f(r; E, \ell)$ from part a) to discuss all possible types of motion. Show that there are qualitative changes compared to Galilean mechanics; in particular that the particle can reach the center provided $\zeta := \alpha^2/\ell^2 c^2 > 1$. Determine the pericenter (distance closest to the center) and the apocenter (distance farthest from the center) and show that in the Keplerian result is recovered in the nonrelativistic limit $\zeta \to 0$. Find the condition for the existence of an allowed region and again show that the Keplerian result is recovered for $\zeta \to 0$.

hint: Write f in the form

$$f(r; E, \ell) = \frac{2\epsilon + \epsilon^2}{(1+\epsilon)^2} - \zeta\,\frac{(p/r)^2}{(1+\epsilon)^2}$$

with $p = \ell^2/m\alpha$ the quantity known as 'parameter' in celestial mechanics and $\epsilon = (E + \alpha/r)/mc^2$ a local energy parameter. Discuss ϵ as a function of r and the two contributions to f as functions of ϵ to obtain a qualitative picture of f as a function of r. Then solve the condition $f(r_\pm; E, \ell) = 0$ to obtain the turning points of the radial motion.

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d) Assume $\zeta < 1$ and determine the equation of orbit. Show that the result can be written in a closed form that generalizes the Galilean result:

$$p^*/r = 1 + e^* \cos(\sqrt{1-\zeta}\,\phi) \quad .$$

Determine p^* and e^* and convince yourself that in the nonrelativistic limit you recover the Keplerian orbit.

e) Use the preceding results to calculate the perihelion advance of Mercury to leading order in $1/c^2$. Estimate the accuracy of that approximation. Compare your result with the experimental result of 43"/century.

hint: The semi-major axis, period, and eccentricity, respectively, of Mercury's Galilean orbit are: $a = 5.79 \times 10^{12}$ cm, $T = 88$ days, and $e = 0.206$

f) Consider the classical motion of an electron in the field of a Th nucleus ($\alpha = Ze^2$, $Z = 90$). Draw the orbit on a scale of $1 : 10^{-10}$ for the four cases $\ell = \hbar, 2\hbar$, $E = \pm 12\,keV$. (Use $\hbar = 10^{-27}$ erg, $\hbar c/e^2 = 137$.) Also draw the corresponding Galilean orbits.

(40 points)