Chapter 2 | Static solutions of Maxwell's equations

§1 Poisson's equation

1.1 Electrostatics

Consider M's eqs for static fields: (2) $\vec{\nabla}\times\vec{E}=0$

(3) $\vec{\nabla}\cdot\vec{E}=4\pi\rho$

remark: (1) (2) and (3) contain $\vec{E}$ only, (1) $\vec{\nabla}\cdot\vec{B}=0$, and (4) $\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{j}$ contain $\vec{B}$ only. → For static fields, $\vec{E}$ and $\vec{B}$ decouple.

ch 1 § 3.4 → A static $\vec{E}$-field is determined by φ alone:

$$\boxed{\vec{E}(\vec{x})=-\vec{\nabla}\varphi(\vec{x})} \quad (*)$$

remark: (2) M eq. (2) is automatically satisfied:

$$(\vec{\nabla}\times\vec{E})_i=-(\vec{\nabla}\times\vec{\nabla}\varphi)_i=-\varepsilon_{ijk}\,\partial_j\partial_k\varphi=0$$

proposition 1: The electrostatic potential φ obeys Poisson's eq.:

$$\boxed{\vec{\nabla}^2\varphi(\vec{x})=-4\pi\rho(\vec{x})} \quad (**)$$

where $\vec{\nabla}^2\equiv\Delta\equiv\partial_i\partial^i$ is the Laplace operator.

corollary. In vacuum, φ obeys the Laplace eq.

$$\boxed{\vec{\nabla}^2\varphi(\vec{x})=0}$$

remark: (2') Solutions of Laplace's eq are called harmonic fcts.

(3) $\varphi(\vec{x})=\text{const}$, $\varphi(\vec{x})=x$, $\varphi(\vec{x})=y$, and $\varphi(\vec{x})=z^2-\frac{1}{2}(x^2+y^2)$ all are harmonic fcts.

(4) A harmonic fct can have no extrema, except at infinity.

[illegible]

1.2 Magnetostatics

ch 1 § 3.4 -> $\vec{B} = \vec{\nabla}\times\vec{A}$

remark: (1) This is always true, but in particular for static fields.

(2) M-eq. (1) static is automatically fulfilled, via $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$.

proposition 1: The Euclidean vector potential $\vec{A}$ obeys

$$\boxed{\vec{\nabla}^2\vec{A}(\vec{x}) = -\frac{4\pi}{c}\vec{j}(\vec{x})} \quad (*)$$

proof: $(\vec{\nabla}\times\vec{\nabla}\times\vec{A})_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_l A_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_l A_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m = -\partial_j\partial_j A_i + \partial_i\partial_m A_m$

$= -\vec{\nabla}^2 A_i + \partial_i(\vec{\nabla}\cdot\vec{A})$

Problem 8 -> We can always choose Coulomb gauge, which makes $\vec{\nabla}\cdot\vec{A} = 0$

-> $\frac{4\pi}{c}\vec{j} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times\vec{\nabla}\times\vec{A} = -\vec{\nabla}^2\vec{A}$ $\square$

remark: (3) The components $(\varphi, \vec{A})$ of the static electromagnetic potential obey Poisson's eq. with $-\frac{4\pi}{c}$ times the components $(c\rho, \vec{j})$ of the 4-current as the inhomogeneity.

(4) Poisson's eq. is linear -> the most general solution is a particular solution plus the most general solution of the Laplace eq. (see ch 1 § 5.1 corollary 2).

(5) § 1.1 remark (4) -> the only solution of Laplace's eq. that vanishes at infinity is the zero solution!
-> In an infinite system, there is only one physical solution of Poisson's eq.

(6) This is not [illegible]

§2 Solutions of Poisson's equation

2.1 The general solution of Poisson's equation

proposition: Every Fourier transformable solution of Poisson's eq. is uniquely determined by the inhomogeneity ρ via

$$\boxed{\varphi(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\vec{x}}\, \frac{4\pi}{k^2}\, \hat{\rho}(\vec{k})} \quad (*)$$

proof: §1.1 (**) $\leadsto$ $-k^2 \hat{\varphi}(\vec{k}) = -4\pi \hat{\rho}(\vec{k})$

$\leadsto$ $\hat{\varphi}(\vec{k}) = 4\pi \hat{\rho}(\vec{k})/k^2$

Fourier backtransform $\leadsto$ (*) $\square$

remark: (1) Thanks to the theory developed in 610 ch 2 §4, the class of solutions that can be constructed in this way is large!

(2) §1.2 remark (5) follows immediately from Fourier theory

$\Delta\varphi(\vec{x}) = 0 \iff -k^2\hat{\varphi}(\vec{k}) = 0 \iff \hat{\varphi}(\vec{k}) = 0 \;\forall\, \vec{k} \neq 0$

$\iff \varphi(\vec{x}) \equiv \text{const.}$

(3) All of this is consistent with §1.2 remark (4).

2.2 The Coulomb potential

Consider a point charge: $\rho(\vec{x}) = e\,\delta(\vec{x})$ where $\delta(\vec{x}) := \delta(x)\delta(y)\delta(z)$

theorem: The electrostatic potential resulting from a point charge is the Coulomb potential $\boxed{\varphi(\vec{x}) = \frac{e}{r}}$ with $r = |\vec{x}|$

proof: $\hat{\rho}(\vec{k}) = e \rightsquigarrow \hat{\varphi}(\vec{k}) = 4\pi e/k^2$

610 Problem 33.b. $\rightsquigarrow$ $\underline{\underline{\varphi(\vec{x}) = e/r}}$ $\square$

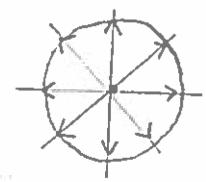
remark: (1) We have now derived the Coulomb potential (from a least-action principle) that we had postulated in PHYS 610.

corollary: The electric field of a point charge is

$$\boxed{\vec{E}(\vec{x}) = e\,\vec{x}/r^3}$$

$\vec{E}_{\vec{k}} \equiv \hat{\vec{E}}(\vec{k})$ etc.
$\hat{k} \equiv \vec{k}/k$

proof: $\underline{\underline{\vec{E}(\vec{x})}} = -\vec{\nabla}\varphi(\vec{x}) = -\vec{\nabla}\,\frac{e}{r} = \underline{\underline{e\,\frac{\vec{x}}{r^3}}}$ $\square$



remark: (2) The $\vec{E}$-field of a point charge is purely radial and isotropic

2.3 Poisson's formula

proposition: Let $\rho(\vec{x})$ be a charge distribution whose Fourier transform exists, then

$$\boxed{\varphi(\vec{x}) = \int d\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}}$$ Poisson's formula

Problem 16: $\vec{E}$-field of ring + disk

Problem 17 spherical charge distributions

proof: §2.1 (a) $\rightsquigarrow$ $\varphi(\vec{x})$ is the Fourier backtransform of the product

$$\varphi_{\vec{k}} = v^c_{\vec{k}}\,\rho_{\vec{k}} \quad \text{with} \quad v^c_{\vec{k}} = 4\pi/k^2$$

But the Fourier backtransform of $4\pi/k^2$ is $1/|\vec{x}|$, and that of $\rho_{\vec{k}}$ is $\rho(\vec{x})$, and by the convolution theorem (610 ch2 §2.1) we have

$$\underline{\underline{\varphi(\vec{x})}} = \int d\vec{y}\, v_c(\vec{x}-\vec{y})\,\rho(\vec{y}) = \underline{\underline{\int d\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho(\vec{y})}}$$ $\square$

Problem 18 $\rightarrow$ d-dim Coulomb problem

remark: (1) For $\rho(\vec{x}) = e\,\delta(\vec{x})$ we recover the ...

$\overset{e}{\otimes}\!\longrightarrow \vec{v}$

2.4 The field of a uniformly moving charge

Consider a charge e that moves with a constant velocity $\vec{v}$ with respect to an observer.

Let CS' be the inertial frame in which the charge is at rest.

§2.2 $\rightsquigarrow$ $\varphi'(\vec{x}') = e/r'$ and $A'^{\mu}(x') = (\varphi'(x'), 0)$

Let CS be the inertial frame of the observer, and let $\vec{v} = (v,0,0)$.

CS and CS' are related by a Lorentz boost in x-direction.

ch 1 §4.1 $\rightsquigarrow$ $x' = \gamma(x - vt)\,,\; y' = y\,,\; z' = z \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

and $$\varphi = \gamma\varphi' = \gamma\frac{e}{r'} = \gamma\frac{e}{\sqrt{x'^2+y'^2+z'^2}}$$

$$= \gamma\frac{e}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{1/2}} = \frac{e}{\left[(x-vt)^2+(1-v^2/c^2)(y^2+z^2)\right]^{1/2}}$$

$\rightsquigarrow$ The scalar potential due to the moving charge is

$$\boxed{\varphi(\vec{x},t) = e/R^*(\vec{x},t)} \quad \text{where} \quad R^*(\vec{x},t) := \left[(x-vt)^2+(1-v^2/c^2)(y^2+z^2)\right]^{1/2} = r'(\vec{x},t)/\gamma$$

and the vector potential is

$$\boxed{\vec{A}(\vec{x},t) = \gamma\frac{\vec{v}}{c}\varphi' = \frac{\vec{v}}{c}\varphi(\vec{x},t) = \frac{e\vec{v}}{cR^*(\vec{x},t)}}$$

Now consider the fields. In CS' we have

$$\vec{E}'(\vec{x}') = e\,\vec{x}'/r'^3\,,\quad \vec{B}'(\vec{x}') = 0$$

ch 1 §4.2 $\rightsquigarrow$ $E_x = E_x' = \dfrac{ex'}{r'^3} = \dfrac{e}{\gamma^2}\dfrac{x-vt}{(R^*(\vec{x},t))^3}\,,\quad E_y = \gamma E_y' = \gamma\dfrac{ey'}{r'^3} = \dfrac{e}{\gamma^2}\dfrac{y}{(R^*(\vec{x},t))^3}\,,\quad E_z = \gamma E_z' = \gamma\dfrac{ez'}{r'^3} = \dfrac{e}{\gamma^2}\dfrac{z}{(R^*(\vec{x},t))^3};$

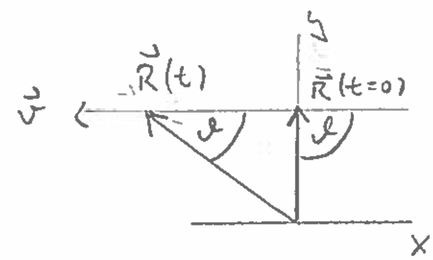
let $\vec{R} = (x - vt, y, z)$ $\rightarrow$ $\boxed{\vec{E}(\vec{x},t) = \frac{e}{\gamma^2} \frac{\vec{R}(t)}{(R^*(\vec{x},t))^3}}$ electric field seen by the observer

cf 1 fig 2 $\rightarrow$ $B_x = B_x' = 0$, $B_y = -\gamma \frac{v}{c} E_z' = -\frac{v}{c} E_z$

$$B_z = \gamma \frac{v}{c} E_y' = \frac{v}{c} E_y$$

$\rightarrow$ $\boxed{\vec{B}(\vec{x},t) = \frac{1}{c} \vec{v} \times \vec{E}(\vec{x},t)}$ magnetic field seen by the observer

discussion of $\vec{E}(\vec{x},t)$:



let θ be the angle between $\vec{v}$ and $\vec{R}$

$\rightarrow$ $\frac{\sqrt{y^2+z^2}}{|\vec{R}|} = \sin\theta$ $\rightarrow$ $y^2 + z^2 = R^2 \sin^2\theta$

$\rightarrow$ $(R^*)^2 = R^2 - \frac{v^2}{c^2}(y^2+z^2) = R^2\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]$

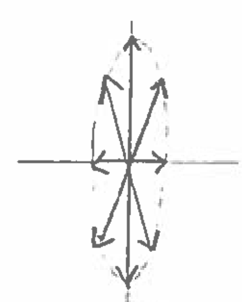
$\rightarrow$ $\boxed{\vec{E}(\vec{x},t) = \frac{e}{\gamma^2} \frac{\vec{R}(\vec{x},t)}{R^3(\vec{x},t)} \frac{1}{\left[1 - \frac{v^2}{c^2}\sin^2\theta(t)\right]^{3/2}}}$ $\vec{R}(\vec{x},t) = (x - vt, y, z)$

$\rightarrow$ For fixed distance R from the charge, $\vec{E}$ is minimal for $\theta = 0, \pi$, i.e., in the direction of the motion. This minimal value is

$$E_{\parallel} = \frac{e}{R^2}(1 - v^2/c^2)$$

$\vec{E}$ is maximal for $\theta = \pm\frac{\pi}{2}$, i.e., in the direction perpendicular to the motion. The maximum value is

$$E_{\perp} = \frac{e}{R^2} \frac{1}{\sqrt{1 - v^2/c^2}}$$



$\rightarrow$ The field is no longer isotropic, but squeezed in the direction of the motion.

remark: (1) Alternative point of view: 4-current in CS' is

$$j'^{\mu} = (c\rho'(x'), \vec{j}'(x')) \quad \text{with} \quad \rho'(x') = e\,\delta(\vec{x}'), \ \vec{j}' = 0$$

→ Observer in CS sees

charge density: $\rho(\vec{x},t) = \gamma\rho'(\vec{x}',t') = \gamma e\,\delta(\gamma(x-vt))\delta(y)\delta(z)$

$$= e\,\delta(x-vt)\delta(y)\delta(z)$$

current density: $\vec{j}(\vec{x},t) = \gamma\frac{\vec{v}}{c}c\rho' = \vec{v}\rho = e\vec{v}\,\delta(x-vt)\delta(y)\delta(z)$

Now solve M's eqs for this time-dependent 4-current.
This is clearly equivalent, but much harder to do!

2.5 Electrostatic interaction

Consider a charge density $\rho(\vec{x})$.

proposition 1: The energy of the electric field produced by $\rho(\vec{x})$ is

$$\boxed{U = \frac{1}{2}\int d\vec{x}\,d\vec{y}\;\rho(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\rho(\vec{y})}$$

proof: $U \hat{=}$ § 1.6 → $U = \frac{1}{8\pi}\int_V d\vec{x}\;\vec{E}^2(\vec{x}) = \frac{-1}{8\pi}\int_V d\vec{x}\;\vec{E}(\vec{x})\cdot\vec{\nabla}\varphi(\vec{x})$

$$= \underbrace{\frac{-1}{8\pi}\int_V d\vec{x}\;\vec{\nabla}\cdot(\vec{E}(\vec{x})\varphi(\vec{x}))}_{=\int_{(V)} d\vec{S}\cdot\vec{E}\varphi \,=\, 0 \text{ for } V\to\infty} + \frac{1}{8\pi}\int_V d\vec{x}\;\varphi(\vec{x})\,\vec{\nabla}\cdot\vec{E}(\vec{x}) \overset{\vec{\nabla}\cdot\vec{E}=4\pi\rho}{=} \frac{1}{2}\int d\vec{x}\;\varphi(\vec{x})\rho(\vec{x})$$

$$\overset{\text{Poisson}}{=} \frac{1}{2}\int d\vec{x}\,d\vec{y}\;\rho(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\rho(\vec{y}).$$

Week 5
Probset 5+6
#16, 17, 18
+ Take-home midterm

remark: (1) Let $\rho(\vec{x})$ be composed of N localized charge distributions: $\rho(\vec{x}) = \sum_{\alpha=1}^{N}\rho^{(\alpha)}(\vec{x})$

$$\leadsto \underline{U} = \frac{1}{2}\sum_{\alpha\beta}\int d\vec{x}\,d\vec{y}\;\rho^{(\alpha)}(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\rho^{(\beta)}(\vec{y})$$

$$= \sum_\alpha U^{(\alpha)} + \sum_{\alpha\neq\beta} U^{(\alpha,\beta)}$$

where $U^{(\alpha)} := \frac{1}{2}\int d\vec{x}\,d\vec{y}\; \rho^{(\alpha)}(\vec{x})\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho^{(\alpha)}(\vec{y})$

"self energy" of the charge distribution

$U^{(\alpha,\beta)} := (1-\delta_{\alpha\beta})\,\frac{1}{2}\int d\vec{x}\,d\vec{y}\; \rho^{(\alpha)}(\vec{x})\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho^{(\beta)}(\vec{y})$

"electrostatic interaction" of localized charge distributions via a Coulomb interaction

(2) Consider charged point particles: $\rho^{(\alpha)}(\vec{x}) = e_\alpha\,\delta(\vec{x}-\vec{x}^{(\alpha)})$

$\rightsquigarrow U^{(\alpha,\beta)} = (1-\delta_{\alpha\beta})\,\frac{1}{2}\,\frac{e_\alpha e_\beta}{|\vec{x}^{(\alpha)}-\vec{x}^{(\beta)}|}$ Coulomb interaction

$U^{(\alpha)}$ does not exist!

(3) The concept of a point charge leads to an infinite self energy and makes no sense within classical electrodynamics. Only the interaction energy of point charges is physically meaningful.

(4) Estimate the smallest spatial extension r_0 of a charge e with mass m that still makes sense:

$$e^2/r_0 \simeq mc^2 \rightsquigarrow r_0 = e^2/mc^2$$

For electrons: $r_0^e = e^2/m_e c^2 = 2.8\times10^{-13}$ cm

"classical electron radius"

Current experimental upper limit on the radius of the electron: $r_e < 10^{-20}$ cm (!)

2.6 The law of Biot and Savart

Problem 19
Helmholtz eq.

proposition 1: A stationary current density distribution $\vec{j}(\vec{x})$ leads to a vector potential

$$\boxed{\vec{A}(\vec{x}) = \frac{1}{c}\int d\vec{y}\, \frac{\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|}}$$

proof: §1.2 $\Rightarrow$ Each component of $\vec{A}$ obeys Poisson's eq $\Rightarrow$ the solution for each component is given by Poisson's formula □

remark: (1) §1.2 proof of prop. 1 $\Rightarrow$ this is true in Coulomb gauge, $\vec{\nabla}\cdot\vec{A}=0$

proposition 2: The magnetic field generated by a stationary current density is

$$\boxed{\vec{B}(\vec{x}) = -\frac{1}{c}\int d\vec{y}\, \frac{(\vec{x}-\vec{y})\times\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|^3}}$$

law of Biot & Savart

proof: $\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})$, and

$$\left(\vec{\nabla}_x \times \frac{\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|}\right)_i = \varepsilon_{ijk}\,\partial_j \frac{j_k(\vec{y})}{\left(\sum_l (x_l-y_l)^2\right)^{1/2}} = \varepsilon_{ijk}\, j_k(\vec{y}) \frac{-1}{2} \frac{2(x_j-y_j)}{|\vec{x}-\vec{y}|^3}$$

$$= -\varepsilon_{ijk}\,(\vec{x}-\vec{y})_j\, j_k(\vec{y}) \frac{1}{|\vec{x}-\vec{y}|^3} = \frac{((\vec{x}-\vec{y})\times\vec{j}(\vec{y}))_i}{|\vec{x}-\vec{y}|^3}$$

remark: (2) Notice the analogy between electrostatics and magnetostatics.

(3) See §3.3 below for a discussion of the concept of a stationary current density.

2.7 Magnetostatic interaction

Consider a current density $\vec{j}(\vec{x})$.

proposition 1: The energy of the magnetic field produced by $\vec{j}(\vec{x})$ is

$$\boxed{U = \frac{1}{2c^2}\int d\vec{x}\, d\vec{y}\; \vec{j}(\vec{x})\, \frac{1}{|\vec{x}-\vec{y}|}\, \vec{j}(\vec{y})}$$

lemma: $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

proof:
$$\underline{\partial_i \varepsilon_{ijk} A_j B_k} = \varepsilon_{ijk}(\partial_i A_j) B_k + \varepsilon_{ijk} A_j \partial_i B_k$$
$$= \underline{B_k \varepsilon_{kij} \partial_i A_j - \varepsilon_{jik} A_j \partial_i B_k}$$

proof of prop:
$$\underline{U} = \frac{1}{8\pi}\int_V d\vec{x}\; \vec{B}^2(\vec{x}) = \frac{1}{8\pi}\int_V d\vec{x}\; \vec{B}\cdot(\vec{\nabla}\times\vec{A})$$
$$\overset{\text{lemma}}{=} \frac{1}{8\pi}\int_V d\vec{x}\; \vec{A}(\vec{x})\cdot\left(\vec{\nabla}\times\vec{B}(\vec{x})\right) + \frac{1}{8\pi}\int_V d\vec{x}\; \vec{\nabla}\cdot\left(\vec{A}(\vec{x})\times\vec{B}(\vec{x})\right)$$
$$\overset{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}}{=} \frac{1}{8\pi}\int_V d\vec{x}\; \vec{A}(\vec{x})\cdot\frac{4\pi}{c}\vec{j}(\vec{x}) + \underbrace{\frac{1}{8\pi}\int_{\partial V} d\vec{\sigma}\cdot\left(\vec{A}(\vec{x})\times\vec{B}(\vec{x})\right)}_{\to 0 \text{ for } V\to\infty}$$
$$= \frac{1}{2c}\int_V d\vec{x}\; \vec{j}(\vec{x})\cdot\frac{1}{c}\int d\vec{y}\; \vec{j}(\vec{y})\,\frac{1}{|\vec{x}-\vec{y}|}$$
$$= \underline{\underline{\frac{1}{2c^2}\int d\vec{x}\, d\vec{y}\; \vec{j}(\vec{x})\cdot\frac{1}{|\vec{x}-\vec{y}|}\cdot\vec{j}(\vec{y})}} \qquad \square$$

remark: (1) For localized current distributions $\vec{j}(\vec{x}) = \sum_\alpha \vec{j}^{(\alpha)}(\vec{x})$ we again distinguish between

$$U^{(\alpha)} = \frac{1}{2c^2}\int d\vec{x}\, d\vec{y}\; \vec{j}^{(\alpha)}(\vec{x})\,\frac{1}{|\vec{x}-\vec{y}|}\cdot\vec{j}^{(\alpha)}(\vec{y}) \quad \text{magnetostatic self energy}$$

and

$$U^{(\alpha,\beta)} = (1-\delta_{\alpha\beta})\frac{1}{2c^2}\int d\vec{x}\, d\vec{y}\; \vec{j}^{(\alpha)}(\vec{x})\,\frac{1}{|\vec{x}-\vec{y}|}\,\vec{j}^{(\beta)}(\vec{y}) \quad \text{magnetostatic interaction}$$

$$\to \underline{U = \sum_\alpha U^{(\alpha)} + \sum_{\alpha\neq\beta} U^{(\alpha,\beta)}}$$

§3 Multipole expansion for static fields

3.1 The electric dipole moment

Q: Given a localized charge distribution $\rho(\vec{\xi})$, what are the potential $\varphi(\vec{x})$ and the field $\vec{E}(\vec{x})$, at a point far from the charges?

Let $\rho(\vec{\xi}) \equiv 0$ for $|\vec{\xi}| > r_0$, let $|\vec{x}| = r \gg r_0$, and write

$$\frac{1}{|\vec{x}-\vec{\xi}|} = \frac{1}{\sqrt{r^2 - 2\vec{x}\cdot\vec{\xi} + \xi^2}} = \frac{1}{r}\Big(1 - 2\underbrace{\frac{\vec{x}\cdot\vec{\xi}}{r^2}}_{=O(r_0/r)} + \underbrace{\frac{\xi^2}{r^2}}_{=O(r_0^2/r^2)}\Big)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \frac{\vec{x}\cdot\vec{\xi}}{r^2} + O(r_0^2/r^2)\right)$$

Poisson's formula (§2.3) $\rightsquigarrow$

$$\varphi(\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \int d\vec{\xi}\, \rho(\vec{\xi})\, \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{\xi}}{r^2} + O(r_0^2/r^2)\right]$$

$$= \frac{1}{r}\int d\vec{\xi}\, \rho(\vec{\xi}) + \frac{\vec{x}}{r^3}\cdot\int d\vec{\xi}\, \vec{\xi}\, \rho(\vec{\xi}) + O(1/r^3)$$

proposition: For large distances r from the localized charge distribution the scalar potential has the form

$$\boxed{\varphi(\vec{x}) = \frac{Q}{r} + \frac{\vec{d}\cdot\vec{x}}{r^3} + O(1/r^3)}$$

where $Q = \int d\vec{\xi}\, \rho(\vec{\xi})$ is the total charge

and $\vec{d} = \int d\vec{\xi}\, \vec{\xi}\, \rho(\vec{\xi})$ is the electric dipole moment

of the charge distribution.

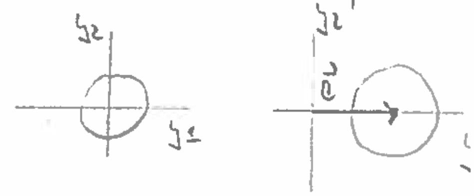
Problem 20
Quadrupole moments

remark: (1) Analogous results hold for the potential of a localized mass distribution, see PHYS 611.

If you are yet confused about this, consider a collection of point charges e_α at locations $\vec{x}_\alpha$: $\rho(\vec{y}) = \sum_\alpha e_\alpha \delta(\vec{y} - \vec{x}_\alpha)$. Shifted ρ means shifted charge locations: $\vec{x}_\alpha{}' = \vec{x}_\alpha + \vec{a}$, and hence $\rho'(\vec{y}) = \sum_\alpha e_\alpha \delta(\vec{y} - \vec{x}_\alpha - \vec{a})$

(2) If $Q = 0$, then the dipole moment is independent of the origin of the coordinate system: let $\vec{x}\,' = \vec{x} + \vec{a}$ with $\vec{a} = \text{const}$. Then in the new coordinate system we have

$$\rho'(\vec{y}) = \rho(\vec{y} - \vec{a})$$



$$\rightarrow \underline{\vec{d}\,'} = \int d\vec{y}\; \vec{y}\, \rho'(\vec{y})$$

$$= \int d\vec{y}\; \vec{y}\, \rho(\vec{y} - \vec{a}) = \int d\vec{y}\, (\vec{y} + \vec{a})\, \rho(\vec{y}) = \vec{d} + \vec{a}\, Q = \underline{\vec{d}}$$

corollary: The field at large distances is

$$\boxed{\vec{E}(\vec{x}) = Q\, \frac{\vec{x}}{r^3} + \frac{3(\hat{x}\cdot\vec{d})\,\hat{x} - \vec{d}}{r^3} + O(1/r^4)} \qquad \hat{x} = \vec{x}/|\vec{x}|$$

proof: $\vec{E} = -\vec{\nabla}\varphi$ and $-\vec{\nabla}\, Q/r = Q\,\vec{x}/r^3$, see §2.2

$$\underline{\vec{\nabla}\, \frac{\vec{d}\cdot\vec{x}}{r^3}} = \frac{1}{r^3}\, \vec{\nabla}(\vec{d}\cdot\vec{x}) + (\vec{d}\cdot\vec{x})\, \vec{\nabla}\frac{1}{r^3}$$

$$= \frac{1}{r^3}\, \vec{d} + (\vec{d}\cdot\vec{x})\left(-\frac{3}{2}\right)\frac{1}{r^5}\, 2\vec{x} = \underline{\frac{\vec{d}}{r^3} - 3\,\frac{1}{r^3}\,(\vec{d}\cdot\hat{x})\,\hat{x}}$$

remark: (3) For $Q = 0$, the leading contribution to the field falls off as $1/r^3$.

(4) Obviously, this expansion can be continued, with the next term being the quadrupole moment (a rank-2 tensor, see PHYS 611$^{\#}$). However, it is advantageous to introduce a more general concept.

$\#$ and Problem #20

Week 6

New assignment

Theorem 1: "Completeness"
Any piecewise continuous and cont. diff'ble fct
$f(x) : [-1,1] \to \mathbb{R}$ can be expanded in Legendre polynomials:

$$\boxed{f(x) = \sum_{\ell=0}^{\infty} f_\ell \, P_\ell(x)}$$

where the coefficients are given by

$$f_\ell = \frac{2\ell+1}{2} \int dx \, f(x) \, P_\ell(x)$$

Note: In W2017 they had just seen this in QM, so I went through it very quickly

3.2 Legendre functions, and spherical harmonics (for props, see Math books)

def. 1: The polynomials of degree l defined by

$$\boxed{P_l(x) := \frac{1}{2^l\, l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l} \qquad l = 0,1,2,\dots$$

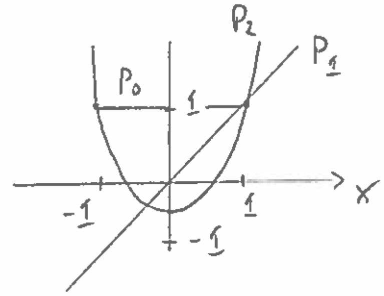
are called Legendre polynomials.

remark: (1) The first few Legendre polynomials are

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$



(2) The $P_l(x)$ have the following properties:

(o) $P_l(1) = 1 \;\forall l$

(i) $P_l(-x) = (-)^l P_l(x)$ parity

(ii) $(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$ diff. eq.

(iii) $(l+1)P_{l+1}(x) = (2l+1)\,x\,P_l(x) - l\,P_{l-1}(x)$ recursion relation

(iv) $\int_{-1}^{1} dx\, P_l(x)\,P_{l'}(x) = \delta_{ll'}\,\frac{2}{2l+1}$ orthogonality

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(3) The Legendre polynomials are a member of a more general family called orthogonal polynomials. See, e.g., Dennery & Krzywicki

def. 2. The functions

$$\boxed{P_l^m(x) := \frac{(-)^m}{2^l\, l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l} \qquad \begin{array}{l} l = 0,1,2,\dots \\ m = -l,-l+1,\dots,l-1,l \end{array}$$

are called associated Legendre functions

remark: (4) $P_l^0(x) = P_l(x)$

(5) For fixed l, there are $2l+1$ P_l^m.

(6) The first few $P_l^m(x)$ are

$$P_0^0(x) = P_0(x) = 1$$

$$P_1^0(x) = P_1(x) = x \,, \quad P_1^1(x) = -\sqrt{1-x^2} \,, \quad P_1^{-1}(x) = \tfrac{1}{2}\sqrt{1-x^2}$$

(7) The P_l^m have the properties

DO!

(i) $P_l^m(x=\pm 1) = 0$ for $m \neq 0$ zeros

(ii) $P_l^{-m}(x) = (-)^m \dfrac{(l-m)!}{(l+m)!} P_l^m(x)$ symmetry

(iii) $\dfrac{d}{dx}\left[(1-x^2)\dfrac{d}{dx}P_l^m(x)\right] + \left[l(l+1) - \dfrac{m^2}{1-x^2}\right]P_l^m(x) = 0$ ODE

(iv) $\displaystyle\int_{-1}^{1} dx\, P_l^m(x)\, P_{l'}^{\overset{\leftarrow \text{same } m!}{m}}(x) = \delta_{ll'}\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$ orthogonality

(v) $(l+1-m)P_{l+1}^m(x) = (2l+1)\,x\,P_l^m(x) - (l+m)P_{l-1}^m(x)$ recursion relations

(vi) $(2l+1)\sqrt{1-x^2}\,P_l^m(x) = P_{l-1}^{m+1}(x) - P_{l+1}^{m+1}(x)$

def. 7: Consider a unit sphere. Let $\Omega = (\vartheta, \varphi)$ be a point on the sphere, and let $\eta = \cos\vartheta$ $(-1 \le \eta \le 1)$. Then the $\mathbb{C}$-valued functions defined on the sphere by

$$\boxed{Y_{lm}(\Omega) = \left[\frac{(2l+1)(l-m)!}{4\pi\,(l+m)!}\right]^{1/2} e^{im\varphi}\, P_l^m(\eta)}$$

are called spherical harmonics.

(7') Different books define the normalization differently!

remark: (8) $Y_{00}(\Omega) = \dfrac{1}{\sqrt{4\pi}}$

$$Y_{10}(\Omega) = \sqrt{\frac{3}{4\pi}}\cos\vartheta \,, \quad Y_{1,\pm 1}(\Omega) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\varphi}\sin\vartheta$$

(9) The Y_{lm} have the properties

(i) $Y_{lm}^*(\Omega) = (-)^m Y_{l,-m}(\Omega)$ complex conjugation

(ii) $-i\dfrac{\partial}{\partial\varphi} Y_{lm}(\Omega) = m\, Y_{lm}(\Omega)$

$\Delta Y_{lm}(\Omega) = -l(l+1)\, Y_{lm}(\Omega)$ ODEs

where $\Lambda = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$
$= \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{1}{1-\eta^2}\frac{\partial^2}{\partial\varphi^2}$ is the angular part of the Laplace operator in spherical coordinates.

(iii) $\int d\Omega\, Y^*_{\ell m}(\Omega)\, Y_{\ell' m'}(\Omega) = \delta_{\ell\ell'}\delta_{mm'}$ orthogonality

theorem 2: Any piecewise continuous and continuously diff'ble fct. on the sphere $f(\Omega)$ can be expanded in terms of spherical harmonics:

$$\boxed{f(\Omega) = \sum_{\ell,m} f_{\ell m} Y_{\ell m}(\Omega)}$$

and the coefficients are given by

$$\boxed{f_{\ell m} = \int d\Omega\, f(\Omega)\, Y^*_{\ell m}(\Omega)}.$$

remark: (10) This is often referred to by saying "the $Y_{\ell m}$ form a complete set on the sphere".

proposition: Addition theorem

let $\Omega = (\vartheta,\varphi)$ and $\Omega' = (\vartheta',\varphi')$ and let γ be the angle between the two points:

$\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi')$

Problem 21
Legendre polynomials
Problem 22
associated Legendre fcts
Problem 23
Spherical harmonics

then

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\Omega')\, Y_{\ell m}(\Omega)}$$

corollary: For $\gamma = 0$ we have $\Omega = \Omega'$ and $P_\ell(1) = 1$

$$\Rightarrow \boxed{\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\Omega)|^2 = \frac{2\ell+1}{4\pi}}$$ "sum rule"

3.3 Separation of the Laplace operator in spherical coordinates

Consider the Laplace operator

$$\vec{\nabla}^2 \equiv \Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\Lambda \quad \text{with} \quad \Lambda = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

from §3.2 remark (a)

$$\rightsquigarrow \quad \Delta f(r,\vartheta,\varphi) = \left(\frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Lambda\right) f(r,\varphi,\vartheta) = \underbrace{\frac{1}{r}\partial_r^2 r}_{\text{acts on } r \text{ only}} f + \underbrace{\frac{1}{r^2}\Lambda}_{\text{acts on } \vartheta,\varphi \text{ only}} f$$

Theorem: The differential eq.

$$\boxed{\left[-\Delta + V(r)\right]\psi(r,\vartheta,\varphi) = a(r,\vartheta,\varphi)} \quad (*) \quad \text{for the fct. } \psi(\vec{x})$$

is solved by

$$\boxed{\psi(r,\vartheta,\varphi) = \sum_{l,m} \frac{1}{r} u_{lm}(r)\, Y_{lm}(\Omega)}$$

where $u_{lm}(r)$ is the solution of the ODE

$$\boxed{\left(-\frac{d^2}{dr^2} + V_l(r)\right) u_{lm}(r) = r\, a_{lm}(r)} \quad (**)$$

where $\quad V_l(r) = V(r) + \frac{l(l+1)}{r^2}$

and $\quad a_{lm}(r) = \int d\Omega\, a(r,\vartheta,\varphi)\, Y^*_{lm}(\Omega) \quad (+)$

remark: (1) The Poisson eq. has the form $(*)$

(2) This theorem is also very useful in QM.

postpone

proof of theorem: ansatz: $\psi(r,\vartheta,\varphi) = \frac{1}{r}\sum_{lm} u_{lm}(r)\, Y_{lm}(\Omega)$

$$(*) \rightsquigarrow \quad -\frac{1}{r}\partial_r^2 r\, \frac{1}{r}\sum_{lm} u_{lm}(r)\, Y_{lm}(\Omega) - \frac{1}{r^3}\sum_{lm} u_{lm}(r)\, \underbrace{\Lambda Y_{lm}(\Omega)}_{= -l(l+1)Y_{lm} \text{ by §3.2 (a)}} + \frac{V(r)}{r}\sum_{lm} u_{lm}(r)\, Y_{lm} = a(r,\Omega)$$

p42/.p

proof: Consider of 3.7 with $V(r)=0$, $c(r,\vartheta,\varphi)=0$

$$\rightsquigarrow \partial_r^2 u_{\ell m}(r) = \frac{\ell(\ell+1)}{r^2} u_{\ell m}(r)$$

ansatz: $u_{\ell m}(r) = r^n \rightsquigarrow n(n-1) = \ell(\ell+1)$

$$\rightsquigarrow n^2 - n - \ell(\ell+1) = 0$$

$$\rightsquigarrow n = \frac{1}{2}\left(1 \pm \sqrt{1+4\ell(\ell+1)}\right) = \frac{1}{2}\left(1 \pm |2\ell+1|\right) = \begin{cases} \ell+1 \\ -\ell \end{cases}$$

$\rightsquigarrow$ the two linearly independent solutions are

$$\varphi^-_{\ell m}(\bar{x}) = \text{const} \times \frac{1}{r} r^{\ell+1} Y_{\ell m}(\Omega) = \text{const} \times r^\ell Y_{\ell m}(\Omega)$$

and $\varphi^+_{\ell m}(\bar{x}) = \text{const} \times \frac{1}{r} r^{-\ell} Y_{\ell m}(\Omega) = \text{const} \times \frac{1}{r^{\ell+1}} Y_{\ell m}(\Omega)$,

§3.2 theorem 2 → any reasonably well behaved $c(r,\Omega)$ can be expanded in spherical harmonics: $c(r,\Omega) = \sum_{lm} c_{lm}(r) Y_{lm}(\Omega)$ with $c_{lm}(r)$ given by (+).

$$\rightarrow \sum_{lm} \left[-\frac{1}{r}\partial_r^2 u_{lm}(r) + \frac{l(l+1)}{r^3} u_{lm}(r) + \frac{V(r)}{r} u_{lm}(r) \right] Y_{lm}(\Omega) = \sum_{lm} c_{lm}(r) Y_{lm}(\Omega)$$

$$\rightarrow \left[-\partial_r^2 + \left(V(r) + \frac{l(l+1)}{r^2} \right) \right] u_{lm}(r) = r\, c_{lm}(r) \qquad \square$$

3.4 Expansion of harmonic fcts in spherical harmonics

Consider harmonic fcts, i.e., solutions of

$$\boxed{\Delta \varphi(\vec{x}) = 0} \quad (*)$$

and assume that φ is twice continuously diff'ble.

proposition: The most general solution of $(*)$ has the form

$$\boxed{\varphi(\vec{x}) = \sum_{lm} \left[\varphi^+_{lm}(\vec{x}) + \varphi^-_{lm}(\vec{x}) \right]}$$

where $\boxed{\varphi^+_{lm}(\vec{x}) = g^+_{lm} \frac{1}{r^{l+1}} Y_{lm}(\Omega)}$

$\boxed{\varphi^-_{lm}(\vec{x}) = g^-_{lm} r^l Y_{lm}(\Omega)}$

with constant coefficients $g^\pm_{lm}$.

remark: (1) $\varphi^+_{lm}(\vec{x}\to 0) \to \infty \ \forall\, l$, $\varphi^-_{lm}(\vec{x}\to\infty) \to \infty \ \forall\, l > 0$

→ The only harmonic fct. that is finite at $r=0$ and at $r=\infty$ is the constant $l=0$ contribution, and the only one that is finite at $r=0$ and zero at $r=\infty$ is

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3.5 Multipole expansion of the electrostatic potential

Lemma:
$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_+}\sum_{l=0}^{\infty}\left(\frac{r_-}{r_+}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\Omega)\,Y_{lm}(\Omega')}$$

where $\vec{x}=(r,\Omega)$, $x'=(r',\Omega')$, $r_+=\max(r,r')$
$r_-=\min(r,r')$

proof: let $\cos\gamma = \vec{x}\cdot\vec{x}'/rr'$

$\rightarrow |\vec{x}-\vec{x}'| = \sqrt{r^2-2rr'\cos\gamma+r'^2}$

1st case: $r>r'$ $\rightarrow$
$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\frac{1}{\left[1-2\frac{r'}{r}\cos\gamma+\left(\frac{r'}{r}\right)^2\right]^{1/2}} = \frac{1}{r_+}\left[1-2\frac{r_-}{r_+}\cos\gamma+\left(\frac{r_-}{r_+}\right)^2\right]^{-1/2}$$

$$\overset{\S 3.2\ \text{theorem}}{=} \frac{1}{r_+}\sum_{l=0}^{\infty} f_l\left(\frac{r_-}{r_+}\right)P_l(\cos\gamma)$$

$$\overset{\S 3.2\ \text{prop}}{=} \frac{1}{r_+}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1} f_l\left(\frac{r_-}{r_+}\right)\sum_{m=-l}^{l} Y_{lm}^*(\Omega')\,Y_{lm}(\Omega)$$

Remaining question: What is $f_l(r_-/r_+)$?

§3.4 $\rightarrow$ $\frac{1}{|\vec{x}-\vec{x}'|}$ is a harmonic fct. for $r>r'$, since
$$\Delta_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|} = \Delta_{\vec{x}}\frac{1}{r} = \frac{1}{r}\partial_r^2 r\frac{1}{r} = 0$$

Furthermore, $\frac{1}{|\vec{x}-\vec{x}'|} = O(1/r)$ for $r\to\infty$

§3.4 $\rightarrow$ $\frac{1}{r}f_l\left(\frac{r'}{r}\right) = \frac{1}{r}\left(\frac{r'}{r}\right)^l c_l$ with some constant c_l

Put $\gamma=0$ $\rightarrow$ $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r}\frac{1}{1-r'/r} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l$ $\rightarrow$ $\underline{c_l = 1}$

$$\rightarrow \underline{f_l\left(\frac{r_-}{r_+}\right) = \left(\frac{r_-}{r_+}\right)^l}$$

2nd case: $r'>r$ analogous $\square$

proposition: The electrostatic potential of a localized charge distribution $\rho(\vec{x})$ ($\rho(\vec{x})=0$ for $|\vec{x}|>r_0$) can be written, for $|\vec{x}|>r_0$,

$$\boxed{\varphi(\vec{x}) = \sum_{l,m} \frac{Q_{lm}}{r^{l+1}} \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{lm}(\Omega)}$$

problem 24: quadrupole moments etc.

where the $Q_{lm} = \left(\frac{4\pi}{2l+1}\right)^{1/2} \int_0^\infty dr\, r^{2+l} \int d\Omega\, \rho(r,\Omega)\, Y_{lm}^*(\Omega)$

are the multipole moments of the charge distribution

proof: §2.3 $\rightsquigarrow$ $\varphi(\vec{x}) = \int d\vec{y}\, \frac{\rho(\vec{y})}{|\vec{y}-\vec{x}|} \overset{|\vec{x}|>r_0}{=} \int d\vec{y}\, \rho(\vec{y}) \frac{1}{r} \sum_{l=0}^\infty \left(\frac{y}{r}\right)^l \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \times Y_{lm}^*(\Omega_y) Y_{lm}(\Omega_x)$

$$= \sum_{l,m} \frac{1}{r^{l+1}} Y_{lm}(\Omega_x) \frac{4\pi}{2l+1} \int_0^\infty dy\, y^{2+l} \int d\Omega_y\, \rho(\vec{y}) Y_{lm}^*(\Omega_y)$$

remark: (1) the $l=0$ moment is

$$Q_{00} = \sqrt{4\pi} \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega) \frac{1}{\sqrt{4\pi}} = Q \quad \text{total charge}$$

and the $l=1$ moments are

$$Q_{1m} = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr\, r^3 \int d\Omega\, \rho(r,\Omega) \sqrt{\frac{3}{4\pi}} \left[\delta_{m,0}\cos\vartheta - \delta_{m,1}\frac{1}{\sqrt{2}} e^{-i\varphi}\sin\vartheta + \delta_{m,-1}\frac{1}{\sqrt{2}} e^{+i\varphi}\sin\vartheta\right]$$

$$\rightsquigarrow Q_{10} = \int_0^\infty dr\, r^2 \int d\Omega\, r\cos\vartheta\, \rho(r,\Omega) = \int d\vec{x}\, x_3\, \rho(\vec{x}) = d_3$$

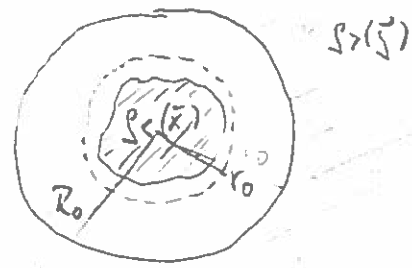
$$Q_{11} = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega)\, r \sqrt{\frac{3}{8\pi}} e^{-i\varphi} (-) \sqrt{1-\xi^2}$$

$$= \frac{-1}{\sqrt{2}} \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega)\, r\sin\vartheta \left[\cos\varphi - i\sin\varphi\right] = \frac{-1}{\sqrt{2}}(d_1 - i d_2)$$

$$Q_{1,-1} = \frac{1}{\sqrt{2}}(d_1 + i d_2) \qquad d_1 = \frac{1}{\sqrt{2}}(Q_{1,-1} - Q_{11})$$

$$d_2 = \frac{-i}{\sqrt{2}}(Q_{1,-1} + Q_{11})$$

3.6 Multipole expansion of the electrostatic interaction



Consider a charge density $\rho_<(\vec{x})$ confined to a region $R_<$ inside a sphere of radius r_0. Let $\rho_<(\vec{x})$ be subject to a field generated by a charge density $\rho_>(\vec{y})$ confined to a region $R_>$ outside a sphere with radius R_0. §2.5 ⇝ The electrostatic interaction energy of this system is

$$\underline{\underline{U}} = \frac{1}{2}\int_{R_<} d\vec{x}\,\rho_<(\vec{x})\int_{R_>} d\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho_>(\vec{y}) + \frac{1}{2}\int_{R_>} d\vec{x}\,\rho_>(\vec{x})\int_{R_<} d\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho_<(\vec{y})$$

$$= \int_{R_<} d\vec{x}\,\rho_<(\vec{x})\int_{R_>} d\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho_>(\vec{y}) = \underline{\underline{\int d\vec{x}\,\rho_<(\vec{x})\,\varphi_>(\vec{x})}}$$

where

$$\underline{\varphi_>(\vec{x})} = \underline{\int_{R_>} d\vec{y}\,\frac{1}{|\vec{x}-\vec{y}|}\,\rho_>(\vec{y})}$$

is the potential generated by the charges in the region $R_>$.

If $R_0 \gg r_0$, $\varphi_>(\vec{x})$ will vary slowly within $R_<$ ⇝ Taylor expand

$$\varphi_>(\vec{x}) = \varphi_>(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\varphi_>\Big|_{\vec{x}=0} + \frac{1}{2}x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}\varphi_>\Big|_{\vec{x}=0} + \dots$$

§1.1 ⇝ $\varphi_>(\vec{x})$ obeys Laplace's eq. $\forall\,\vec{x}\in R_<$

$$\leadsto \delta_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\varphi_>\Big|_{\vec{x}=0} = 0$$

$$\leadsto \varphi_>(\vec{x}) = \varphi_>(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\varphi_>\Big|_{\vec{x}=0} + \frac{1}{2}\left(x_i x_j - \frac{\vec{x}^2}{3}\delta_{ij}\right)\frac{\partial^2}{\partial x_i \partial x_j}\varphi_>\Big|_{\vec{x}=0} + \dots$$

<u>def. 1</u>: Denote by
$\varphi_0 := \varphi_>(\vec{x}=0)$ the potential $\varphi_>$ at the origin
$\vec{E} := -\vec{\nabla}\varphi_>(\vec{x}=0)$ the field due to $\varphi_>$ at the origin
$\varphi_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}\varphi_>(\vec{x}=0)$ the field gradient at the origin

$$\rightarrow \quad \varphi_>(\bar{x}) = \varphi_0 - \vec{x}\cdot\vec{E} + \frac{1}{2}\left(x_i x_j - \frac{\vec{x}^2}{3}\delta_{ij}\right)\varphi_{ij} + \dots$$

Now drop the < on $\rho_<$ and the > on $\varphi_>$ and write

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$$\underline{U} = \int d\vec{x}\, \rho(\bar{x})\,\varphi(\bar{x}) = \varphi_0 \int d\vec{x}\, \rho(\bar{x}) - \vec{E}\cdot\int d\vec{x}\, \vec{x}\,\varphi(\bar{x}) + \frac{1}{3}\varphi_{ij}\,\frac{1}{2}\int d\vec{x}\,(3x_i x_j - \delta_{ij}\vec{x}^2)\,\rho(\bar{x}).$$

Problem 25
field due to distant charges

$$\rightarrow \quad \boxed{U = \varphi_0 Q - \vec{E}\cdot\vec{d} + \frac{1}{3}\varphi^{ij} Q_{ij} + \dots}$$

where $\varphi_0, \vec{E}, \varphi^{ij}$ are the potential, electric field, and field gradient tensor due to $\rho_>$ evaluated at the origin

Problem 26
charged spheroid
in a Parallelepiped

and $Q, \vec{d}, Q_{ij}$ are the total charge, dipole moment, and quadrupole moments of $\rho_<$.

remark: (1) Alternatively, we can use §3.4 to expand

$$\varphi(\vec{x}) = \sum_{lm} \varphi_{lm}^{-}(\bar{x}) = \sum_{lm} q_{lm}^{-}\, r^l\, Y_{lm}(\Omega)$$

$$\rightarrow \underline{\underline{U}} = \int d\vec{x}\, \varphi(\bar{x})\,\rho(\bar{x}) = \int d\vec{x} \sum_{lm} q_{lm}^{-}\, r^l\, Y_{lm}(\Omega)\,\rho(\bar{x})$$

$$= \sum_{lm} q_{lm}^{-} \int_0^\infty dr\, r^{2+l} \int d\Omega\, Y_{lm}(\Omega)\,\rho(\bar{x})$$

$$\overset{§3.5}{=} \sum_{lm} q_{lm}^{-} \left(\frac{2l+1}{4\pi}\right)^{1/2} Q_{lm}^{*}$$

where the Q_{lm} are the multipole moments of the charge density $\rho(\bar{x}) \equiv \rho_<(\bar{x})$ and the q_{lm}^{-} are the coefficients of the expansion of the harmonic fct. $\varphi(\bar{x}) = \varphi_>(\bar{x})$ in spherical harmonics.

Week 7
Worksheet 7 (§19,20,21,22)

3.7 The magnetic moment

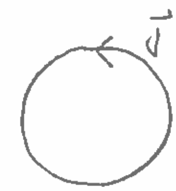
§2.6 $\leadsto$ The law of Biot & Savart gives the magnetic field resulting from a stationary current distribution. This requires an interpretation as currents are produced by moving charges and hence are intrinsically time dependent.

def 1: By a stationary current density $\vec{j}(\vec{x})$ we mean the time average taken over a time T that is very long compared to all microscopic time scales:

$$\vec{j}(\vec{x}) := \overline{\vec{j}(\vec{x},t)} \equiv \frac{1}{T}\int_0^T dt\, \vec{j}(\vec{x},t)$$

example: (1) Current in a wire loop.

$T \gg$ time it takes an electron to complete one revolution.



remark: (1) With this definition the 4th Maxwell eq. reduces to its static version upon time averaging, provided the electric field $\vec{E}$ as a function time is bounded:

$$\overline{\partial \vec{E}(\vec{x},t)/\partial t} = \frac{1}{T}\int_0^T dt\, \frac{\partial \vec{E}}{\partial t} = \frac{1}{T}\left[\vec{E}(\vec{x},T) - \vec{E}(\vec{x},0)\right] \xrightarrow{T\to\infty} 0$$

if $\vec{E}(\vec{x},t)$ is bounded.

$$\leadsto -\frac{1}{c}\overline{\partial_t \vec{E}} + \overline{\vec{\nabla}\times\vec{B}} = \boxed{\overline{\vec{\nabla}\times\vec{B}} = \frac{4\pi}{c}\vec{j}}$$

Now consider the vector potential $\vec{A}(\vec{x}) \equiv \overline{\vec{A}(\vec{x},t)}$ at large distance from a localized static current density $\vec{j}(\vec{y}) = \overline{\sum_\alpha e_\alpha \vec{v}_\alpha \delta(\vec{y}-\vec{x}_\alpha)}$

$$\S 21.6 \leadsto \underline{\vec{A}(\vec{x})} = \frac{1}{c}\int d\xi \, \frac{\vec{j}(\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \frac{1}{c}\sum_\alpha \frac{e_\alpha \vec{v}_\alpha}{|\vec{x}-\vec{x}_\alpha|}$$

$$\overset{\S 7.1}{=} \frac{1}{c}\sum_\alpha e_\alpha \vec{v}_\alpha \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{x}_\alpha}{r^2} + \dots\right]$$

$$\sum_\alpha e_\alpha \vec{v}_\alpha = \frac{d}{dt}\sum_\alpha e_\alpha \vec{x}_\alpha = 0 \text{ by remark (1)}$$

$$= \frac{1}{c}\frac{1}{r^3}\sum_\alpha e_\alpha \vec{v}_\alpha (\vec{x}_\alpha\cdot\vec{x})$$

consider

$$\sum_\alpha e_\alpha \vec{v}_\alpha (\vec{x}_\alpha\cdot\vec{x}) = \sum_\alpha e_\alpha \dot{\vec{x}}_\alpha (\vec{x}_\alpha\cdot\vec{x}) =$$

$$= \frac{1}{2}\frac{d}{dt}\sum_\alpha e_\alpha \vec{x}_\alpha (\vec{x}_\alpha\cdot\vec{x}) + \frac{1}{2}\sum_\alpha e_\alpha \Big(\vec{v}_\alpha(\vec{x}_\alpha\cdot\vec{x}) - \vec{x}_\alpha(\vec{v}_\alpha\cdot\vec{x})\Big)$$

$$\xrightarrow{\text{time average}} 0 + \frac{1}{2}\sum_\alpha e_\alpha \Big(\vec{v}_\alpha(\vec{x}_\alpha\cdot\vec{x}) - \vec{x}_\alpha(\vec{v}_\alpha\cdot\vec{x})\Big)$$

$$\leadsto \underline{\vec{A}(\vec{x})} = \frac{1}{2c}\frac{1}{r^3}\sum_\alpha e_\alpha \Big(\vec{v}_\alpha(\vec{x}_\alpha\cdot\vec{x}) - \vec{x}_\alpha(\vec{v}_\alpha\cdot\vec{x})\Big)$$

<u>def. 2</u>: The <u>magnetic moment</u> of the charges is defined as

$$\boxed{\vec{m} := \frac{1}{2c}\sum_\alpha e_\alpha (\vec{x}_\alpha \times \vec{v}_\alpha)} = \frac{1}{2c}\int d\vec{x}\; \vec{x}\times\vec{j}(\vec{x})$$

<u>proposition 1</u>: The vector potential for large distances is given by the magnetic moment via

$$\boxed{\vec{A}(\vec{x}) = \frac{1}{r^3}\,\vec{m}\times\vec{x}}$$

<u>proof</u>: $\vec{m}\times\vec{x} = \frac{1}{2c}\sum_\alpha e_\alpha (\vec{x}_\alpha\times\vec{v}_\alpha)\times\vec{x} = \frac{1}{2c}\sum_\alpha e_\alpha \big[\vec{v}_\alpha(\vec{x}_\alpha\cdot\vec{x}) - \vec{x}_\alpha(\vec{v}_\alpha$

corollary 1: The magnetic field for large distances is

$$\boxed{\vec{B}(\vec{x}) = \frac{3(\hat{x}\cdot\vec{m})\hat{x}-\vec{m}}{r^3} + O(1/r^4)}$$

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proof: $\underline{B_i} = (\vec{\nabla}\times\vec{A})_i = \varepsilon_{ijk}\,\partial_j \frac{1}{r^3}\varepsilon_{klm} m_l x_m = \partial_j \frac{1}{r^3}(m_i x_j - m_j x_i) = \frac{1}{r^3}(3m_i - m_i) + (m_i x_j - m_j x_i)\partial_j \frac{1}{r^3}$

$= \frac{2m_i}{r^3} + (m_i x_j - m_j x_i)(-3)\frac{1}{r^5}x_j = \frac{1}{r^3}(2m_i - 3m_i + 3(\vec{m}\cdot\vec{x})x_i/r^2) = \underline{\frac{1}{r^3}[3(\vec{m}\cdot\hat{x})\hat{x}-\vec{m}]}$

proposition 2: If all of the moving charges have the same charge-to-mass ratio $e_\alpha/m_\alpha \equiv e/m$, and if the motion is nonrelativistic, $v_\alpha \ll c$, then the magnetic moment is proportional to the angular momentum $\vec{L}$ of the moving charges:

$$\boxed{\vec{m} = \frac{e}{2mc}\vec{L}} \qquad (*)$$

proof: $\vec{L} = \sum_\alpha \vec{x}_\alpha\times\vec{p}_\alpha = \sum_\alpha m_\alpha\,\vec{x}_\alpha\times\vec{v}_\alpha$

$\Rightarrow \underline{\vec{m}} = \frac{1}{2c}\sum_\alpha e_\alpha(\vec{x}_\alpha\times\vec{v}_\alpha) = \frac{1}{2c}\sum_\alpha \frac{e_\alpha}{m_\alpha} m_\alpha(\vec{x}_\alpha\times\vec{v}_\alpha) = \underline{\frac{e}{2mc}\vec{L}}$

remark: (2) The proportionality factor $\frac{e}{2mc}$ is called gyromagnetic ratio.

(3) (*) holds for the orbital angular momentum $\vec{L}$ of classical particles, but not for the magnetic moment related to the spin of quantum particles. For electrons, $\vec{m}_e = g\,\frac{e}{2m_e c}\vec{S}_e$

with $\vec{S}_e = \frac{1}{2}\hbar$ the spin of the electron and $g = 2.0023\ldots$ the g-factor.

(4) The Dirac eq. yields $g=2$; $g\neq 2$ is due to loop corrections.

p50 f.p.

Note (my eyes only):

This is a very tricky and worrying point. Jackson discusses it, but in a disjointed way. See pp 186, 216 in his book for the magnetostatic case, and pp 142, 161 for the electrostatic one. LL II discuss the electrostatic case in §42, but they refuse to discuss the magnetostatic analog in Vol II! They do discuss the problem in Vol. VIII, where they point out that the roles of the thermodynamic potentials they call F and $\tilde{F}$, respectively, is reversed in the magnetic case compared to the electric one. See Vol. VIII §§31, 32, and especially the remark at the end of §30 in my German edition. 3/4/16

3.8 The magnetostatic energy of a current distribution

Consider the magnetostatic analog of §3.6, i.e., a collection of localized current distributions $j_>(\vec y)$ that are "far away" and another one, $j_<(\vec x)$, that are "close in".

§2.7 → the energy of the magnetic fields that result from these currents is

$$\underline{U} = \frac{1}{c^2}\int d\vec x\, d\vec y\; \vec j_<(\vec x)\cdot\int d\vec y\,\frac{1}{|\vec x-\vec y|}\,\vec j_>(\vec y)$$

$$\overset{§2.6}{=} \underline{\frac{1}{c}\int d\vec x\; \vec j_<(\vec x)\cdot\vec A_>(\vec x)}$$

with $\vec A_>$ the potential generated by the currents $\vec j_>$.

Taylor expand $\vec A_>(\vec x)$ in analogy to §3.6:

$$A_>^i(\vec x) = A_>^i(\vec x=0) + x_j\,\partial^j A_>^i(\vec x)\Big|_{\vec x=0} + \dots$$

$$\to \underline{U} = \frac{1}{c}\int d\vec x\; j_i(\vec x)\,A^i(\vec x=0) + \frac{1}{c}\int d\vec x\; j_i(\vec x)\,x_j\,\partial^j A^i(\vec x)\Big|_{\vec x=0} + \dots$$

$$\int d\vec x\;\vec j(\vec x) = \int d\vec x\,\sum_\alpha e_\alpha\,\overline{\vec v_\alpha\,\delta(\vec x-\vec x_\alpha)} = \sum_\alpha e_\alpha\overline{\vec v_\alpha} = 0 \text{ by §3.7}$$

$$= \frac{1}{c}\int d\vec x\,\sum_\alpha \overline{e_\alpha v_\alpha^i\,\delta(\vec x-\vec x_\alpha)\,x^j}\,\partial_j\Big|_{\vec x=0} A_i(\vec x) + \dots$$

$$= \frac{1}{c}\sum_\alpha \overline{e_\alpha v_\alpha^i x_\alpha^j}\,(\partial_j A_i)_{\vec x=0} =$$

same argument as in §3.7, p48 $\overset{?}{=}$ $\frac{1}{2c}\sum_\alpha e_\alpha\left[\overline{v_\alpha^i x_\alpha^j}\,(\partial_j A_i)_{\vec x=0} - \overline{x_\alpha^i v_\alpha^j}\,(\partial_j A_i)_{\vec x=0}\right]$

$$= \frac{1}{2c}(\partial_j A_i)_{\vec x=0}\sum_\alpha e_\alpha\left(\overline{v_\alpha^i x_\alpha^j - v_\alpha^j x_\alpha^i}\right)$$

Now consider

$$\underline{\vec{B}\cdot\vec{m}} = (\vec{\nabla}\times\vec{A})\cdot\vec{m} = \varepsilon_{ijk}\,\partial_j A_k \,\frac{1}{2c}\sum_\alpha e_\alpha\, \varepsilon_{ilm}\, x_\alpha^l v_\alpha^m$$

$$= \frac{1}{2c}(\partial_j A_k)\sum_\alpha e_\alpha (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\, x_\alpha^l v_\alpha^m$$

$$= \underline{\frac{1}{2c}(\partial_j A_k)\sum_\alpha e_\alpha (x_\alpha^j v_\alpha^k - v_\alpha^j x_\alpha^k)}$$

$$\rightarrow \boxed{U = \vec{B}\cdot\vec{m} + (\text{quadrupole term})}$$

where $\vec{B}$ is the field due to $\vec{j}_>$ evaluated at the origin
$\vec{m}$ is the magnetic moment of the $\vec{j}_<$

remark: (1) This has the opposite sign of the dipole term in the corresponding electrostatic expansion, §2.6 !

(2) This is not the energy of a magnetic dipole with fixed moment $\vec{m}$ in an external field $\vec{B}$ (which we know is $-\vec{m}\cdot\vec{B}$, ∀). Rather, it is the energy of the total field configuration resulting from the $\vec{j}_>$ and $\vec{j}_<$, which includes the work that was done to get the currents flowing. That is, the time dependent processes hidden in the time average necessary to even talk about a static current density leads to a fundamental difference between the magnetostatic energy and the electrostatic one.

∀ see also §3.9

3.9 The energy of dipoles in external fields

To find the energy of a fixed magnetic dipole (e.g., an electron spin) in a magnetic field, consider the force exerted by a field $\vec{B}(\vec{x})$ (produced by $\vec{j}_>$) on a current distribution $\vec{j}$ ($=\vec{j}_<$)

ch 1 §§ 3.5a, 3.5 $\Rightarrow$ the magnetic or Lorentz force on a single charge is $\frac{e}{c}\vec{v}\times\vec{B}$ $\Rightarrow$ the force on $\vec{j}(\vec{x}) = \sum_\alpha e_\alpha \vec{v}_\alpha \delta(\vec{x}_\alpha - \vec{x})$ is

$$\vec{F}_{mag} = \frac{1}{c}\int d\vec{x}\, \vec{j}(\vec{x})\times\vec{B}(\vec{x}) \overset{\text{Taylor}}{=} \frac{1}{c}\int d\vec{x}\, \vec{j}(\vec{x})\times\left[\vec{B}(\vec{x}=0) + (\vec{x}\cdot\vec{\nabla})\big|_{\vec{x}=0}\vec{B}(\vec{x}) + \ldots\right.$$

Lemma: (1) $\int d\vec{x}\, \vec{j}(\vec{x}) = 0$

(2) $\int d\vec{x}\, (x_i j_j(\vec{x}) + x_j j_i(\vec{x})) = 0$

(3) $\int d\vec{x}\, (\vec{a}\cdot\vec{x})\vec{j}(\vec{x}) = -\frac{1}{2}\vec{a}\times\int d\vec{x}\,(\vec{x}\times\vec{j}(\vec{x}))$ with $\vec{a}=$ const

proof: $\vec{j}(\vec{x}) = \sum_\alpha e_\alpha \vec{v}_\alpha \delta(\vec{x}-\vec{x}_\alpha)$

$\Rightarrow$ (1) $\int d\vec{x}\, \vec{j}(\vec{x}) = \sum_\alpha e_\alpha \overline{\vec{v}_\alpha} = \frac{d}{dt}\overline{\sum_\alpha e_\alpha \vec{x}_\alpha} = 0$ by §3.7 model

(2) $\int d\vec{x}\,(x_i j_j + x_j j_i) = \sum_\alpha e_\alpha \overline{(x_\alpha^i v_\alpha^j + x_\alpha^j v_\alpha^i)} = \frac{d}{dt}\overline{\sum_\alpha e_\alpha x_\alpha^i x_\alpha^j} = 0$

by the same argument

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(3) $\int d\vec{x}\,(\vec{a}\cdot\vec{x})j_i(\vec{x}) = \int d\vec{x}\, a_j x_j j_i \overset{(2)}{=} \frac{1}{2}\int d\vec{x}\, a_j (x_j j_i - x_i j_j)$

$= -\frac{1}{2}\varepsilon_{ijk} a_j \int d\vec{x}\, \varepsilon_{klm} x_l j_m = -\frac{1}{2}\left(\vec{a}\times\int d\vec{x}\,(\vec{x}\times\vec{j})\right)_i$

$$\Rightarrow \vec{F}^i_{mag} = -\frac{1}{c}\Big(\vec{B}(\vec{x}=0)\times\underbrace{\int d\vec{x}\,\vec{j}(\vec{x})}_{=0 \text{ by lemma (1)}}\Big)_i + \frac{1}{c}\varepsilon_{ijk}\int d\vec{x}\, j_j(\vec{x})(\vec{\nabla}B_k)_{\vec{x}=0}\cdot\vec{x} + \ldots$$

$$\overset{\text{lemma (3)}}{=} -\frac{1}{2c}\varepsilon_{ijk}\left((\vec{\nabla}B_k)(\vec{x}=0)\times\int d\vec{x}\,(\vec{x}\times\vec{j}(\vec{x}))\right)_j + \ldots$$

§3.7 def ?

$$\overset{\downarrow}{=} -\epsilon_{ijk}\left((\vec{\nabla} B_k)(\vec{x}=0)\times\vec{m}\right)_j + \dots = \epsilon_{ijk}\left((\vec{m}\times\vec{\nabla}|_{\vec{x}=0}) B_k\right)_j + \dots$$

$$\Rightarrow \underline{\underline{\vec{F}_{mag}}} = (\vec{m}\times\vec{\nabla})|_{\vec{x}=0}\times\vec{B} + \dots = \vec{\nabla}(\vec{m}\cdot\vec{B}) - \vec{m}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) + \dots = \vec{\nabla}|_{\vec{x}=0}(\vec{m}\cdot\vec{B}(\vec{x}))$$

$$= \underline{\underline{\vec{\nabla} U}} \quad \text{with } U \text{ from §3.8}$$

That $\vec{F}$ is the gradient of minus the desired potential energy

$$\Rightarrow \vec{F}_{mag} = -\vec{\nabla}\tilde{U}$$

remark: (0) Here we interpret $\tilde{U}$ as $\vec{x}$-dependent via $\vec{B}$: $\tilde{U}(\vec{x}) = -\vec{m}\cdot\vec{B}(\vec{x})$ and put $\vec{x}=0$ after taking the gradient.

where $\boxed{\tilde{U} = -U = -\vec{m}\cdot\vec{B} + \dots}$ is the potential energy of the magnetic dipole $\vec{m}$ in the magnetic field $\vec{B}$

remark: (1) In a QM context, with $\vec{m}$ the magnetic moment of a spin, this is often referred to as the Zeeman energy.

(2) For the corresponding electrostatic problem, the force exerted by an electric field $\vec{E}$ on a charge distribution $\rho(\vec{x})$ is given by

$$\boxed{\vec{F}_{el} = -\vec{\nabla} U} \quad \text{with } U \text{ from §3.6, see Problem 27}$$

This is a fundamental difference between electrostatics and magnetostatics!

Problem 27

Electric charges in an external field

Week 8

Probset 8 (#23, 24, 25)