Chapter 5 Some Aspects of the Electrodynamics of Continuous Media

§1 Maxwell equations for a dielectric medium

1.1 Electrostatics of dielectrics

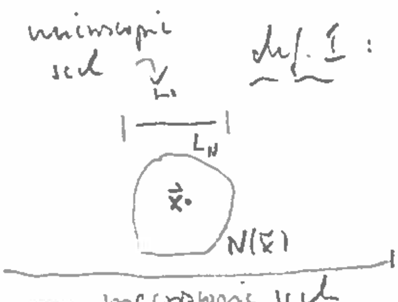
Consider the third Maxwell eq. from ch I, §3.2:

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = 4\pi\rho(\vec{x}),$$

and consider a dielectric, i.e., an electric insulator.

remark: (1) $\rho(\vec{x})$ is a complicated function that varies rapidly on microscopic length scales.

(2) For macroscopic observations we are not interested in these variations.



def. 1: We define a coarse-grained charge density $\bar{\rho}(\vec{x})$ by

$$\bar{\rho}(\vec{x}) = \frac{1}{V_N}\int_{\vec{y}\in N(\vec{x})} d\vec{y}\;\rho(\vec{y})$$

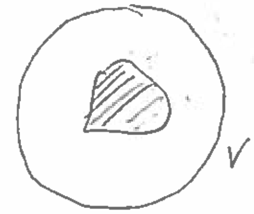
where $N(\vec{x})$ is a neighborhood of the point $\vec{x}$ whose volume V_N is large on the microscopic scale, but small on the macroscopic one.

remark: (3) Microscopic scale ~ Å, macroscopic scale ~ cm → this makes sense.

(4) We also need to coarse-grain the electric field, denote that by $\vec{E}$ as well ($\bar{\vec{E}} \to \vec{E}$; we call the microscopic field $\vec{e}$ and write $\bar{\vec{e}} = \vec{E}$)

def. 2: Define the dielectric polarization field $\vec{P}(\vec{x})$ by

$$\vec{P}(\vec{x}) = \begin{cases} \text{the solution of } -\vec{\nabla}\cdot\vec{P}(\vec{x}) = \bar{\rho}(\vec{x}) & \text{inside the dielectric bo} \\ 0 & \text{outside the dielectric bo} \end{cases}$$



remark: (5) Let V be any volume that completely encloses the dielectric body. Then

$$\int_V d\vec{x}\, \rho(\vec{x}) = 0$$

and $\int_V d\vec{x}\, \vec{\nabla}\cdot\vec{P}(\vec{x}) = \int_{\partial V} d\vec{\sigma}\cdot\vec{P}(\vec{x}) = 0$ ✓

def. 3: $\boxed{\vec{D}(\vec{x}) := \vec{E}(\vec{x}) + 4\pi\vec{P}(\vec{x})}$ is called electric induction

theorem: The coarse-grained 3rd Maxwell eq. takes the form

$$\boxed{\vec{\nabla}\cdot\vec{D}(\vec{x}) = 0} \quad (*)$$

proof: $\vec{\nabla}\cdot\vec{E} = 4\pi\rho = -4\pi\vec{\nabla}\cdot\vec{P}$
$\rightsquigarrow \vec{\nabla}\cdot(\vec{E}+4\pi\vec{P}) = 0$ ∘

remark: (6) Suppose the dielectric is not neutral, but carries an "external" charge density $\rho_{ext}(\vec{x})$ that has been applied by the experimentalist. Then $(*)$ gets generalized to

$$\boxed{\vec{\nabla}\cdot\vec{D}(\vec{x}) = 4\pi\rho_{ext}(\vec{x})} \quad (*')$$

(7) Consider the dipole moment of the dielectric:

$$d_i = \int_V d\vec{x}\, x_i\, \rho(\vec{x}) = -\int_V d\vec{x}\, x_i\, \partial_j P_j = -\int_V d\vec{x}\, \partial_j(x_i P_j) + \int d\vec{x}\, \delta_{ij} P_j$$

$$= \underbrace{-\int_{\partial V} d\sigma_j\, P_j\, x_i}_{=0 \text{ since } \vec{P}=0 \text{ outside the body}} + \int d\vec{x}\, P_i \quad \rightsquigarrow \quad \vec{d} = \int d\vec{x}\, \vec{P}(\vec{x})$$

$\rightsquigarrow$ The polarization is the dipole moment density of the dielectric

Coarse graining the second Maxwell eq, ch. I §1.2, yields

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{x}) = 0} \quad (**)$$

remark: (8) In order for $(*)$ and $(**)$ to give a complete description, we still need a relation between $\vec{D}$ and $\vec{E}$.

remark: (9) $\vec{P}$ is the dipole moment density induced by $\vec{E}$ → $\vec{P}$ must vanish as $\vec{E} \to 0$

assumption: For small $\vec{E}$, and in an isotropic medium, $\vec{P} \propto \vec{E}$:

$$\boxed{\vec{P}(\vec{x}) = \chi(\vec{x})\,\vec{E}(\vec{x})}$$

with χ the dielectric susceptibility.

remark: (10) χ characterizes the medium. In a homogeneous isotropic medium it is a simple number; in a crystal it is a tensor: $P_i = \chi_{ij} E_j$

Problem 49
polarization of a dielectric sphere

(11) Our ignorance about the microscopic details is hidden in χ.

For the relation between $\vec{D}$ and $\vec{E}$ this implies (see def. I)

$$\boxed{\vec{D} = \varepsilon\,\vec{E}}$$

Week 8
Problems 17 (46, 47 pt 2, 48) with $\boxed{\varepsilon = 1 + 4\pi\chi}$ the dielectric constant

1.2 Magnetostatics

Now consider the first Maxwell eq. from ch. I § 3.2:

$$\vec{\nabla}\cdot\vec{B}(\vec{x}) = 0$$

Coarse grain, and call the coarse-grained magnetic induction $\vec{B}$ again →

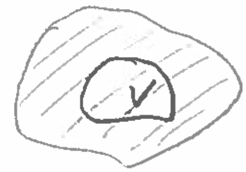
$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{x}) = 0}$$

And the fourth Maxwell eq. for static fields:

$$\vec{\nabla}\times\vec{B}(\vec{x}) = \frac{4\pi}{c}\,\vec{j}(\vec{x})$$

with $\overline{\vec{j}(\vec{x})} = \frac{1}{V_N} \int_{\vec{y} \in N(\vec{x})} d\vec{y}\, \vec{j}(\vec{y})$ the coarse-grained current density.

remark: (1) In a dielectric, no current can flow



$$\to \int_{(V)} d\vec{\sigma} \cdot \vec{j}(\vec{x}) = 0$$
$$\int_{V} d\vec{x}\, \vec{\nabla} \cdot \vec{j}(\vec{x})$$

where the integration is over an arbitrary subvolume of the dielectric $\to \vec{\nabla} \cdot \vec{j}(\vec{x}) = 0$

$\to \vec{j}(\vec{x})$ must be a pure curl!

def. 1: Define the magnetization field $\vec{M}(\vec{x})$ by

$$\vec{M}(\vec{x}) = \begin{cases} \text{the solution of } \vec{\nabla} \times \vec{M}(\vec{x}) = \frac{1}{c} \overline{\vec{j}(\vec{x})} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

def. 2: Define the magnetic field $\vec{H}(\vec{x})$ by

$$\vec{H}(\vec{x}) := \vec{B}(\vec{x}) - 4\pi \vec{M}(\vec{x})$$

theorem: The coarse-grained fourth Maxwell eq. reads

$$\vec{\nabla} \times \vec{H}(\vec{x}) = 0$$

proof: $\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{B} - 4\pi \vec{\nabla} \times \vec{M} = \vec{\nabla} \times \vec{B} - \frac{4\pi}{c} \vec{j} = 0$ □

remark: (2) Since we are discussing dielectrics, there can be no "external currents".

assumption: For small $\vec{H}$, in an isotropic medium,

$\vec{M}(\vec{x}) = \chi_m \vec{H}(\vec{x})$ with χ_m the magnetic susceptibility

For the relation between $\vec{B}$ and $\vec{H}$ this implies

$$\vec{B} = \vec{H} + 4\pi\vec{M} = \vec{H} + 4\pi\chi_m \vec{H} \quad \rightsquigarrow$$

$$\boxed{\vec{B} = \mu\vec{H}}$$

with $\boxed{\mu = 1 + 4\pi\chi_m}$ the <u>magnetic permeability</u>

<u>1.3 Summary of static Maxwell eqs in a dielectric medium</u>

§§ 1.1, 1.2 ⇝

$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{x}) = 0} \quad (1) \qquad \boxed{\vec{\nabla}\cdot\vec{D}(\vec{x}) = 4\pi\rho_{ext}(\vec{x})} \quad (3)$$

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{x}) = 0} \quad (2) \qquad \boxed{\vec{\nabla}\times\vec{H}(\vec{x}) = 0} \quad (4)$$

with $\vec{E}$, $\vec{D}$ the coarse-grained electric field and the electric induction, respectively,

and $\vec{B}$, $\vec{H}$ the coarse-grained magnetic induction and the magnetic field, respectively

and $\boxed{\vec{D} = \varepsilon\vec{E}\ ,\quad \vec{B} = \mu\vec{H}}$ (*)

<u>Problem 5D</u>
Solve eq. in a dielectric medium

with ε the dielectric constant and μ the magnetic permeability.

<u>remark</u>: (1) The constitutive relations (*) are based on a <u>linear</u> relation between $\vec{D}$ and $\vec{E}$, and between $\vec{B}$ and $\vec{H}$. This is approximately true for small fields. For strong fields (e.g., lasers), ε becomes $\vec{E}$-dependent.

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1.4 Generalization to dynamics: Retarded dielectric response

§1.1 → For static fields, $\vec{P} = \chi \vec{E}$

remark: (1) In what follows we will ignore the spatial dependence of the fields, and of χ

For a time-dependent field, the polarization $\vec{P}$ cannot follow the field $\vec{E}$ instantaneously → the assumption from §1.1 needs to be generalized:

assumption:

$$\vec{P}(t) = \int dt' \, \Theta(t>t') \, f(t-t') \, \vec{E}(t')$$

remark: (1) Now a function $f(t)$ characterizes the medium, instead of a single number χ. It is sometimes called a memory function

(2) We still assume that the relation between $\vec{P}$ and $\vec{E}$ is linear ("linear response")

(3) The step function ensures causality.

§2 Introduction to the theory of causal functions

2.1 Causal functions (NO pp 19-21)

Let $f: \mathbb{R} \to \mathbb{R}$ be a slowly increasing fct. in the sense of 610/§4.3. (ch 2)

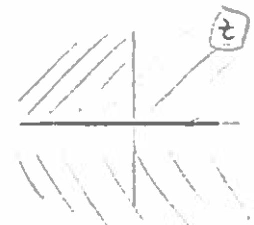
def. 1: $$\boxed{f_\pm(t) := \Theta(\pm t)\, f(t)}$$ are called the retarded (+) and advanced (−) part of f.

remark: (1) PH&S 610 → Fourier trafos of f, f_+, and f_- exist; they are generalized fcts. We denote them by $\hat{f}$, $\hat{f}_\pm$.

def. 2: Let $z \in \mathbb{C}$ and define the Laplace trafo of $f(t)$, i.e.,

$$\boxed{F(z) := \pm i \int dt\, \Theta(\pm t)\, e^{izt} f(t)} \quad \pm \text{ for } \operatorname{Im} z \gtrless 0$$

F is called the causal fct. associated with f. z is called complex frequency.

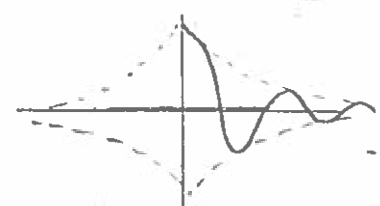


remark: (2) $z = \omega \pm i\delta$ with $\delta > 0$

$$\to e^{izt} = e^{i\omega t} e^{\mp \delta t} = e^{i\omega t} e^{-\delta(\pm t)}$$

→ $F(z)$ is an analytic fct. of z for $\operatorname{Im} z \neq 0$

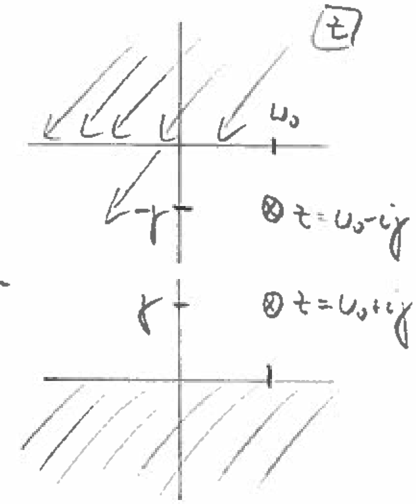
example: (1) $f(t) = e^{-i\omega_0 t} e^{-\gamma |t|}$. damped oscillation



$$i\int_0^\infty dt\, e^{izt - i\omega_0 t - \gamma t} = \frac{-i}{iz - i\omega_0 - \gamma} = \frac{-1}{z - \omega_0 + i\gamma}$$

$$-i\int_{-\infty}^0 dt\, e^{izt - i\omega_0 t + \gamma t} = \frac{-i}{iz - i\omega_0 + \gamma} = \frac{-1}{z - \omega_0 - i\gamma}$$

$$\to \boxed{F(z) = \frac{-1}{z - \omega_0 + i\gamma\, \operatorname{sgn} \operatorname{Im} z}}$$



F(t) is indeed analytic in the upper half plane, and can be analytically continued into the lower half plane where it has a pole at $t=\omega_0-i\gamma$.

Similarly, t is analytic in the lower half plane and its analytic continuation into the upper half plane has a pole at $t=\omega_0+i\gamma$

F(t) consists of two Riemann sheets. It has a branch cut on the real axis, and

$$\underline{\underline{F(\omega\pm i0)}} = \frac{-1}{\omega-\omega_0\pm i\gamma} = \underline{\underline{\frac{-(\omega-\omega_0)\pm i\gamma}{(\omega-\omega_0)^2+\gamma^2}}}$$

Discontinuity of Im F on the real axis (if one stays on the sheet on which the fct is analytic in the respective half plane)

Theorem: The discontinuity of F(t) across the real axis determines the Fourier transform of f(t):

$$\boxed{\lim_{\epsilon\to 0} F(\omega\pm i\epsilon) = \pm i\hat{f}_\pm(\omega)}$$

proof: let Im $t \geq \epsilon > 0$, i.e., $t = x+iy$ with $x,y\in\mathbb{R}$, $y\geq\epsilon$

$\to \quad |e^{itt}| = |e^{ixt}e^{-yt}| \leq e^{-\epsilon t}$

$\to \quad |F(t)| = \left|i\int_0^\infty dt\, e^{itt} f(t)\right| \leq \int_0^\infty dt\, e^{-\epsilon t} f(t) < \infty$ (f(t) weakly increasing)

$\to$ F(ω+iε) is bounded and analytic

$\to$ F(ω+iε) is a regular generalized fct. in the sense of PHYS 610 ch2 §4.4

Now let $\hat{g}(\omega)$ be a test fct., $g \in \mathcal{S}$, in the sense of 610 ℓ2 §4

and consider

(Parseval's eq. 610 ℓ2 §4.3)

$$\int d\omega\, F(\omega + i\varepsilon)\, \hat{g}^*(\omega) = 2\pi i \int_0^\infty dt\, e^{-\varepsilon t} f(t)\, g^*(-t)$$

$$\xrightarrow{\varepsilon \to 0} 2\pi i \int_0^\infty dt\, f(t)\, g^*(-t)$$

But $\left| f(t) e^{-\varepsilon t} g^*(-t) \right| \leq |f(t)| \cdot |g^*(-t)|$

→ the integral exists $\forall \varepsilon$ → $\lim_{\varepsilon \to \infty}$ and $\int dt$ commute

→ $\lim_{\varepsilon \to 0} \int d\omega\, F(\omega + i\varepsilon)\, \hat{g}^*(\omega) = i \int d\omega\, \hat{f}_+(\omega)\, \hat{g}^*(\omega) \quad \forall g \in \mathcal{S}$

→ $\lim_{\varepsilon \to 0} F(\omega + i\varepsilon) = \hat{f}_+(\omega)$ in the sense of 610 ℓ2 §4.4

Analogous proof for $\varepsilon < 0$. □

corollary:

$$\hat{f}(\omega) = \hat{f}_+(\omega) + \hat{f}_-(\omega) = -i\left[F(\omega + i0) - F(\omega - i0)\right]$$

and

$$\hat{f}_+(\omega) - \hat{f}_-(\omega) = -i\left[F(\omega + i0) + F(\omega - i0)\right]$$

example: (2) Consider example (1) with $\gamma \to 0$. → $f(t) = e^{-i\omega_0 t}$

→ $\hat{f}(\omega) = 2\pi\, \delta(\omega - \omega_0)$ and $F(t) = \frac{-1}{t - \omega_0}$

→ $-i\left[F(\omega + i0) - F(\omega - i0)\right] = \lim_{\varepsilon \to 0} \frac{1}{i}\left(\frac{-1}{\omega - \omega_0 + i\varepsilon} + \frac{1}{\omega - \omega_0 - i\varepsilon}\right)$

$$= \lim_{\varepsilon \to 0} \frac{-2i\varepsilon}{i} \frac{-1}{(\omega - \omega_0)^2 + \varepsilon^2} = \lim_{\varepsilon \to 0} \frac{2\varepsilon}{(\omega - \omega_0)^2 + \varepsilon^2} = 2\pi\, \delta(\omega - \omega_0)$$

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2.2 Spectrum and reactive part of causal functions

def. 1: Let $F(z)$ be a causal function. We define

$$F''(\omega) := \frac{1}{2i}\left[F(\omega+i0) - F(\omega-i0)\right]$$

$$F'(\omega) := \frac{1}{2}\left[F(\omega+i0) + F(\omega-i0)\right]$$

F'' is called spectrum, or spectral part, or dissipative part
F' is called reactive part of $F(z)$.

remark: (1) Our notation emphasizes that $f(t), \hat{f}(\omega), F(z), F''(\omega), F'(\omega)$ are different fcts. One often writes $f(t), f(\omega), f(z), f''(\omega), f'(\omega)$ and distinguishes the fcts only by their arguments and the prime and double prime superscripts.

(2) $F''(\omega)$ is given by the discontinuity of $F(z)$ across the real axis; $F'(\omega)$ by the average of $F(z)$ across the real axis.

(3) $F(\omega \pm i0) = F'(\omega) \pm i F''(\omega)$

(4) $\hat{f}(\omega) = \hat{f}_+(\omega) + \hat{f}_-(\omega) = 2F''(\omega)$

$\hat{\hat{f}}(\omega) := \hat{f}_+(\omega) - \hat{f}_-(\omega) = 2F'(\omega)$ ← $(-i)$??

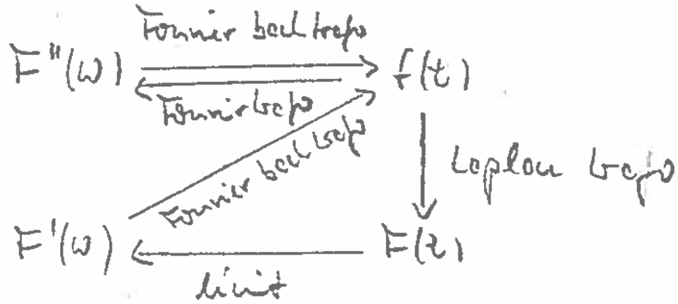
Week 9
hw set 18 (48, 49, 50, 51)

(5) In general, $F'(\omega)$ and $F''(\omega)$ are $\mathbb{C}$-valued fcts.

theorem: (a) The spectrum $F''(\omega)$ uniquely determines $f(t)$, $F(z)$ and $F'(\omega)$

(b) $F'(\omega)$ uniquely determines $F(z)$ and $F''(\omega)$.

remark: (6) This is an extremely important result, even though it follows immediately from the definition proof:



remark: (7) We need the entire fct. $F''(\omega)$, i.e., $F''(\omega)\ \forall \omega$, in order to determine $F'(\omega)$ for a given ω.

corollary: For a given generalized fct. $F''(\omega)$, there exists at most one fct. $F(z)$ such that

(i) $F(z)$ is analytic for all z with $\operatorname{Im} z \neq 0$ and

(ii) $F(z) \to 0$ for $\operatorname{Im} z \to \infty$.

proof: let F_I, F_{II} be two such fcts and consider $G = F_I - F_{II}$

$\rightsquigarrow$ $G(z)$ is analytic for $\operatorname{Im} z \neq 0$ and $G(z) \to 0$ for $\operatorname{Im} z \to \infty$

But $$G(\omega+i0) - G(\omega-i0) = F_I(\omega+i0) - F_{II}(\omega+i0) - F_I(\omega-i0) + F_{II}(\omega-i0)$$

$$= 2i\left[F_I''(\omega) - F_{II}''(\omega)\right] = 2i\left[F''(\omega) - F''(\omega)\right] = 0$$

$\rightsquigarrow$ $G(z)$ is analytic $\forall z$ (the cut contributions cancel!)

A theorem in complex analysis $\rightsquigarrow$ $G(z)$ is a polynomial in z

But $G(z) \to 0$ for $\operatorname{Im} z \to \infty$ $\rightsquigarrow$ $G(z) \equiv 0$ □

2.3 Hilbert - Stieltjes transformation

def. 1: let $f: \mathbb{R} \to \mathbb{C}$ be a generalized fct. such that

$$F(z) := \int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega - z}$$

exists for $\operatorname{Im} z \neq 0$

then $F(z)$ is called the Hilbert - Stieltjes trafo of $f(\omega)$.

example: (1) $f(\omega) = \pi \delta(\omega - \omega_0) \to F(z) = \frac{1}{\omega_0 - z}$

remark: (1) $f \to F$ is an integral trafo that maps $f: \mathbb{R} \to \mathbb{C}$ onto $F: \mathbb{C} \to \mathbb{C}$.

(2) $F(z)$ is analytic for $\operatorname{Im} z \neq 0$ and has a branch cut for $\operatorname{Im} z = 0$

proposition 1: (a) If $f(\omega)$ is even (odd), then $F(z)$ is odd (even)

(b) If $f(\omega) \to F(z)$, then $f^*(\omega) \to F(z^*)^*$

(c) If $f(\omega) \in \mathbb{R}$, then $F(z) = F(z^*)^*$

If $f(\omega) \in i\mathbb{R}$, then $F(z) = -F(z^*)^*$

proof: (a) let $f(\omega) = \pm f(-\omega) \to F(-z) = \int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega + z} = -\int \frac{d\omega}{\pi} \frac{f(-\omega)}{\omega - z}$

$= \mp F(z)$

(b) $\int \frac{d\omega}{\pi} \frac{f(\omega)^*}{\omega - z} = \left(\int \frac{d\omega}{\pi} \frac{f(\omega)}{(\omega - z^*)} \right)^* = F(z^*)^*$

(c) let $f(\omega) \in \mathbb{R} \to F(z^*) = \int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega - z^*} = \left(\int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega - z} \right)^* = F(z)^*$

let $f(\omega) \in i\mathbb{R} \to F(z^*) = \int \frac{d\omega}{\pi} \frac{-f(\omega)^*}{\omega - z^*} = -F(z)^*$

$\square$

theorem 1: (Sokhotski – Plemelj)

If $F(z)$ is bounded for $|\text{Im}\, z| \geq \varepsilon$, then

$$\lim_{\varepsilon \to 0} \frac{1}{2i} \left[F(\omega + i\varepsilon) - F(\omega - i\varepsilon) \right] = f(\omega)$$

simplified proof: let $g(\omega)$ be a test fct. then

$$\frac{1}{2i} \int d\omega\, g(\omega) \left[F(\omega+i\varepsilon) - F(\omega - i\varepsilon) \right] = \frac{1}{2i} \int d\omega\, g(\omega) \int \frac{dx}{\pi} f(x) \left(\frac{1}{x-\omega-i\varepsilon} - \frac{1}{x-\omega+i\varepsilon} \right)$$

$$= \frac{1}{2i} \int d\omega\, g(\omega) \int \frac{dx}{\pi} f(x) \frac{2i\varepsilon}{(x-\omega)^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0} \int d\omega\, g(\omega) \int \frac{dx}{\pi} f(x)\, \delta(x-\omega)$$

$$= \int d\omega\, g(\omega) f(\omega) \quad \forall g \qquad \square$$

theorem 2: let $F: \mathbb{C} \to \mathbb{C}$ be a fct with the properties

(1) $F(z)$ is analytic for $\text{Im}\, z \neq 0$

(2) $F(z) \to 0$ for $|z| \to \infty$

(3) $\lim_{\varepsilon \to 0} \frac{1}{2i} \left[F(\omega + i\varepsilon) - F(\omega - i\varepsilon) \right]$ exists and defines a generalized fct. $f(\omega)$.

Then $F(z)$ can be written in H-S form, i.e., $F(z) = \int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega - z}$ and $F(z)$ is the unique H-S transfo of $f(\omega)$.

proof: books

proposition 2: let $F(z)$ be the H-S transfo of a real generalized fct. $f(\omega) \not\equiv 0$. Then

$$f(\omega) \geq 0 \quad \text{iff} \quad \left(\text{Im}\, F(z) \gtrless 0 \text{ for } \text{Im}\, z \gtrless 0 \right)$$

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proof: $F(x+iy) = \int \frac{d\omega}{\pi} \frac{f(\omega)}{\omega-(x+iy)} = \int \frac{d\omega}{\pi} \frac{(\omega-x+iy)f(\omega)}{(\omega-x)^2+y^2}$

$$= \underbrace{\int \frac{d\omega}{\pi} \frac{(\omega-x)f(\omega)}{(\omega-x)^2+y^2}}_{\in \mathbb{R}} + iy \underbrace{\int \frac{d\omega}{\pi} \frac{f(\omega)}{(\omega-x)^2+y^2}}_{\geq 0 \text{ for } f \geq 0}$$

Therefore, $f(\omega) \geq 0$ implies $\mathrm{Im}\left(\frac{1}{y} F(x+iy)\right) \geq 0$

and $f(\omega) \not\equiv 0$ implies $\mathrm{Im}\left(\frac{1}{y} F(x+iy)\right) > 0$

But $y = \mathrm{Im}\, z \Rightarrow$ $f(\omega) \geq 0$ implies $\mathrm{Im}\, F(z) \gtrless 0$ for $\mathrm{Im}\, z \gtrless 0$

Now let $\mathrm{Im}\, F(z) \gtrless 0$ for $\mathrm{Im}\, z \gtrless 0$

Then $f(\omega) = \frac{1}{2i}\left[F(\omega+i0) - F(\omega-i0)\right] = \frac{1}{2}\left[\mathrm{Im}\, F(\omega+i0) - \mathrm{Im}\, F(\omega-i0)\right] \underset{}{\geq 0}$

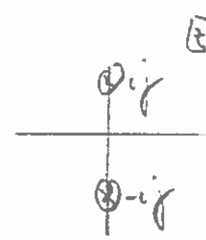
def. 2: The H-S transforms of non-negative fcts are called *positive functions*.

remark: (1) Positivity in this sense is equivalent to $\mathrm{Im}\, F(z) \gtrless 0$ for $\mathrm{Im}\, z \gtrless 0$.

2.4 Simple examples of Hilbert-Stieltjes transforms

(1) simple resonance

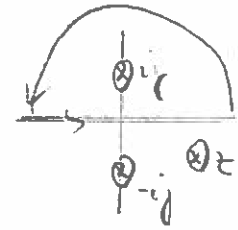
$F(z) = \frac{-1}{z \pm i\gamma}$ for $\mathrm{Im}\, z \gtrless 0$ is analytic for $\mathrm{Im}\, z \neq 0$ and $\to 0$ for $|z| \to \infty$



and $\frac{1}{2i}\left[F(\omega+i0) - F(\omega-i0)\right] = \frac{1}{2i}\left(\frac{-1}{\omega+i\gamma} + \frac{1}{\omega-i\gamma}\right) = \frac{\gamma}{\omega^2+\gamma^2}$ exists

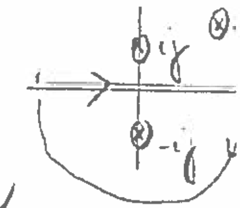
$\Rightarrow \frac{-1}{z \pm i\gamma} = \int \frac{d\omega}{\pi} \frac{1}{\omega - z} \frac{\gamma}{\omega^2+\gamma^2}$

check: $\int \frac{d\omega}{\pi} \frac{1}{\omega - z} \frac{\gamma}{\omega^2+\gamma^2} = \int \frac{d\omega}{\pi} \frac{1}{\omega - z} \frac{\gamma}{(\omega+i\gamma)(\omega-i\gamma)}$



$$= \gamma \frac{2\pi i}{\pi} \frac{1}{2i\gamma} \frac{1}{i\gamma - z} = \frac{-1}{z - i\gamma} \quad \text{for } \mathrm{Im}\, z < 0$$

$$\text{or } \gamma \frac{-2\pi i}{\pi} \frac{1}{-2i\gamma} \frac{1}{-i\gamma - z} = \frac{-1}{z + i\gamma} \quad \text{for } \mathrm{Im}\, z > 0$$

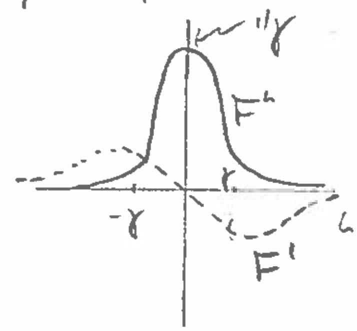


✓

remark: (1) §2.2 → The spectra and the reactive part of $F(z)$ are

$$F''(\omega) = \frac{\gamma}{\omega^2+\gamma^2}$$

$$F'(\omega) = \frac{1}{2}\left(\frac{-1}{\omega+i\gamma} - \frac{1}{\omega - i\gamma}\right) = \frac{-\omega}{\omega^2+\gamma^2}$$



2.5 Spectral representation, and Kramers-Kronig relations

Combining the results of the previous paragraphs, we have the following

theorem: A causal fct. $F(z)$ can be written in terms of its spectrum $F''(\omega)$

$$\boxed{F(z) = \int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega - z}} \quad (*)$$

remark: (1) $(*)$ is called spectral representation or Lehmann representation of $F(z)$.

(2) For $z = \omega' \pm i\varepsilon$ the denominator takes the form

$$\frac{1}{\omega - \omega' \mp i\varepsilon} = \frac{\omega-\omega' \pm i\varepsilon}{(\omega-\omega')^2+\varepsilon^2} = \frac{\omega-\omega'}{(\omega-\omega')^2+\varepsilon^2} \pm i \frac{\varepsilon}{(\omega-\omega')^2+\varepsilon^2}$$

(3) We know that $\lim_{\varepsilon \to 0} \frac{\varepsilon}{(\omega-\omega')^2+\varepsilon^2} = \pi \delta(\omega - \omega')$ is a generalized

def. 1: $\lim_{\varepsilon\to 0} \frac{x}{x^2+\varepsilon^2}$ is called the principal-value generalized fct

and we write $\int dx\, f(x) \lim_{\varepsilon\to 0} \frac{x}{x^2+\varepsilon^2} =: \mathcal{P}\!\int dx\, \frac{f(x)}{x}$

remark: (4) One can show that this exists for large classes of fcts. $f(x)$, and one often writes $\int dx$ instead of $\mathcal{P}\!\int dx$.

corollary: The spectrum and the reactive part of the causal fct. $F(z)$ are related by the Kramers-Kronig relations

$$F'(\omega) = \int \frac{dx}{\pi} \frac{F''(x)}{x-\omega} \quad , \quad F''(\omega) = -\int \frac{dx}{\pi} \frac{F'(x)}{x-\omega}$$

proof: $F'(\omega) = \frac{1}{2}\left[F(\omega+i0)+F(\omega-i0)\right] = \frac{1}{2}\int \frac{dx}{\pi} F''(x)\left(\frac{1}{x-\omega-i0}+\frac{1}{x-\omega+i0}\right)$

$$= \frac{1}{2}\int \frac{dx}{\pi} F''(x)\, \frac{2(x-\omega)}{(x-\omega)^2+0^2} = \mathcal{P}\!\int \frac{dx}{\pi} \frac{F''(x)}{x-\omega}$$

Now consider $\hat{F}(t) = \mathrm{sgn}\,(\mathrm{Im}\, t)\, F(t)$. The premises of §2.3 theorem 2 are fulfilled $\leadsto$ $\hat{F}(t)$ can be written in HS form, and $\hat{F}(\omega\pm i0) = \pm F(\omega\pm i0) \overset{§2.2}{=} \pm F'(\omega) + iF''(\omega)$

$\leadsto \hat{F}''(\omega) = \frac{1}{2i}\left[\hat{F}(\omega+i0)-\hat{F}(\omega-i0)\right] = \frac{1}{i} F'(\omega)$

and $\hat{F}'(\omega) = \frac{1}{2}\left[\hat{F}(\omega+i0)+\hat{F}(\omega-i0)\right] = i F''(\omega)$

$\leadsto F''(\omega) = -i \hat{F}'(\omega) = -i\int \frac{dx}{\pi} \frac{\hat{F}''(x)}{x-\omega} = -\int \frac{dx}{\pi} \frac{F'(x)}{x-\omega}$ □

2.6 Application: The dielectric function

§1.4 ~> The linear relation between the polarization $\vec{P}$ and the electric field $\vec{E}$ is given by a fct. $f_+(t)$ ~> All results of §2 apply!

Laplace trafo ~> $\vec{P}(t) = \chi(t)\,\vec{E}(t)$

with $\chi(t)$ a causal fct.

~> $\chi(\omega \pm i0) = \chi'(\omega) \pm i\chi''(\omega)$ where χ', χ'' obey Kramers-Kronig

§1.1 ~> $\varepsilon(t) = 1 + 4\pi\chi(t)$

~> $\varepsilon'(\omega) = 1 + 4\pi\chi'(\omega)$, $\varepsilon''(\omega) = 4\pi\chi''(\omega)$

KK ~>

$$\varepsilon'(\omega) = 1 + 4\pi\chi'(\omega) = 1 + 4\pi \int \frac{dx}{\pi} \frac{\chi''(x)}{x-\omega} = 1 + \int \frac{dx}{\pi} \frac{\varepsilon''(x)}{x-\omega}$$

$$\varepsilon''(\omega) = 4\pi\chi''(\omega) = -4\pi \int \frac{dx}{\pi} \frac{\chi'(x)}{x-\omega} = \int \frac{dx}{\pi} \frac{1-\varepsilon'(x)}{x-\omega}$$

Kronig 1926

Kramers 192

Remark: (1) symmetry considerations ~> $\varepsilon'(\omega)$ real and even

$\varepsilon''(\omega)$ real and odd

↑ This is confusing! Check how $\chi(t)$ is defined in this context! Is the i in the LTS' trafo missing in the definition!?

example: Measured $\varepsilon(\omega)$ of SiN films

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Infrared dielectric properties of low-stress silicon nitride

Giuseppe Cataldo,[1,2,*] James A. Beall,[3] Hsiao-Mei Cho,[3] Brendan McAndrew,[1] Michael D. Niemack,[3] and Edward J. Wollack[1]

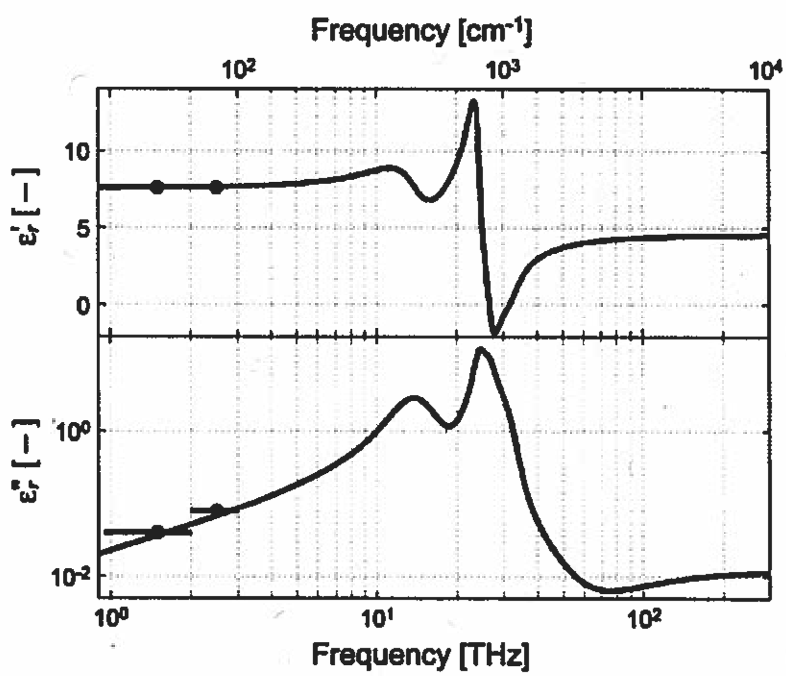

Fig. 3. (Color online) Real and imaginary parts (solid red curves) of the dielectric function as extracted from the data