# Classical Electrodynamics

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# Contents

# Acknowledgments

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## Disclaimer

These notes are still a work in progress. If you notice any mistakes, whether it's trivial typos or conceptual problems, please send email to dbelitz@uoregon.edu.

# Chapter 1

# Mathematical preliminaries

## 1 Vector spaces and tensor spaces

### 1.1 Vector spaces

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$.

We say that $V$ has a set of *basis vectors*

$$\{\boldsymbol{e}_j ; j = 1, \ldots, n\}$$

and *elements* (that is, *vectors*) $\boldsymbol{x}$ expanded in this basis as[1]

$$\boldsymbol{x} = \sum_{j=1}^{n} x^j \boldsymbol{e}_j =: x^j \boldsymbol{e}_j \quad , \quad x^j \in \mathbb{R},$$

where the scalars $x^j$ are called the *coordinates* or *components* of $\boldsymbol{x}$.

**Example 1.** Define the set

$$\mathbb{R}^n := \mathbb{R} \times \cdots \times \mathbb{R} = \left\{ \left(x^1, \ldots, x^n\right) ; x^j \in \mathbb{R} \right\}.$$

$\mathbb{R}^n$ constitutes a vector space over $\mathbb{R}$ if vector addition and scalar multiplication are defined to be the standard real vector addition and real scalar multiplication.

Furthermore, the *Cartesian basis* is

$$\{\boldsymbol{e}_1 = (1, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \boldsymbol{e}_n = (0, \ldots, 0, 1)\}.$$

*Remark* 1. Two vector spaces $V$ and $W$ over the same field $F$ are said to be *isomorphic*, denoted $V \cong W$, iff there exists a bijection $T : V \to W$ that preserves addition and scalar multiplication. That is,

$$\begin{aligned} T(\boldsymbol{x} + \boldsymbol{y}) =& T(\boldsymbol{x}) + T(\boldsymbol{y}), \text{ and} \\ T(c\boldsymbol{x}) =& cT(\boldsymbol{x}) \end{aligned}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and all $c \in F$.[2]

*Claim* 1. All $n$-dimensional vector spaces over $\mathbb{R}$ are isomorphic to $\mathbb{R}^n$.

[1] I have adopted the notation that vectors are bold.

[2] Here, and throughout this document, one must be mindful of what type of variable and what type of operation is written, because often the same symbols are used for addition between vectors and addition between scalars. In this case, $\boldsymbol{x} + \boldsymbol{y}$ is vector addition, $c\boldsymbol{x}$ is scalar-vector multiplication, $T(\boldsymbol{x}) + T(\boldsymbol{y})$ is scalar addition, and $cT(\boldsymbol{x})$ is scalar-scalar multiplication.

*Proof.* In fact, *all* finite-dimensional vector spaces of the same dimension and over the same field are isomorphic to one another. See Theorem 9 of this document. □

## 1.2 Tensor spaces

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ with basis $\{\boldsymbol{e}_j\}$.

**Definition 1.** ***Linear forms.*** A mapping $f : V \to \mathbb{R}$ is called a *linear form* iff

$$\begin{aligned} &\text{(i)} \quad f(\boldsymbol{x}+\boldsymbol{y}) &&= f(\boldsymbol{x}) + f(\boldsymbol{y}) \\ &\text{(ii)} \quad f(c\boldsymbol{x}) &&= cf(\boldsymbol{x}) \end{aligned}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and all $c \in \mathbb{R}$.

**Definition 2.** ***Bilinear forms.*** A mapping $f : V \times V \to \mathbb{R}$ is called a *bilinear form* iff

$$\begin{aligned} &\text{(i)} \quad f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) &&= f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z}) \\ &\text{(ii)} \quad f(\boldsymbol{x}, \boldsymbol{y}+\boldsymbol{z}) &&= f(\boldsymbol{x}, \boldsymbol{y}) + f(\boldsymbol{x}, \boldsymbol{z}) \\ &\text{(iii)} \quad f(c\boldsymbol{x}, \boldsymbol{y}) &&= cf(\boldsymbol{x}, \boldsymbol{y}) \\ &\text{(iv)} \quad f(\boldsymbol{x}, c\boldsymbol{y}) &&= cf(\boldsymbol{x}, \boldsymbol{y}) \end{aligned}$$

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$ and all $c \in \mathbb{R}$.

**Definition 3.** ***Bilinear form components.*** The scalars $t_{jk} := f(\boldsymbol{e}_j, \boldsymbol{e}_k)$ are called the *coordinates* or *components* of the bilinear form $f$ in the basis $\{\boldsymbol{e}_j\}$.

**Proposition 1.** *The coordinates completely determine a bilinear form.*

*Proof.* Let $\boldsymbol{x}, \boldsymbol{y} \in V$. Then

$$f(\boldsymbol{x}, \boldsymbol{y}) = f\left(x^j \boldsymbol{e}_j, y^k \boldsymbol{e}_k\right) = x^j y^k f(\boldsymbol{e}_j, \boldsymbol{e}_k) = t_{jk} x^j y^k.$$

and we see that knowledge of $\{t_{jk}\}$ implies knowledge of $f(\boldsymbol{x}, \boldsymbol{y})$.[a] □

[a] Notice the importance of $f$ obeying properties (i)-(iv) of a bilinear form.

**Definition 4.** ***2-tensors.*** The $n^2$ scalars $t_{jk}$ are called the coordinates of the *rank-2 tensor* (or *2-tensor*) $t$ (which is equivalent to the bilinear form $f$).

*Claim* 1. ***Symmetric forms.*** A bilinear form, $f$, is symmetric if and only if the components of the tensor with respect to the given basis are symmetric; that is,

$$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x}) \ \forall \boldsymbol{x}, \boldsymbol{y} \in V \quad \Leftrightarrow \quad t_{jk} = t_{kj} \ \forall j, k = 1, \ldots, n$$

*Proof.* Assume $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})\ \forall \boldsymbol{x}, \boldsymbol{y} \in V$. Then

$$t_{jk} := f(\boldsymbol{e}_j, \boldsymbol{e}_k) = f(\boldsymbol{e}_k, \boldsymbol{e}_j) = t_{kj}\ \forall j, k = 1, \ldots, n.$$

Now assume $t_{jk} = t_{kj}\ \forall j, k = 1, \ldots, n$. Then

$$f(e_j, e_k) = f(e_k, e_j)\ \forall j, k = 1, \ldots, n.$$

Let $\boldsymbol{x}, \boldsymbol{y} \in V$. These can be expanded as $\boldsymbol{x} = x^j \boldsymbol{e}_j$ and $\boldsymbol{y} = y^j \boldsymbol{e}_j$. Thus,

$$\begin{aligned} x^j y^k f(\boldsymbol{e}_j, \boldsymbol{e}_k) &= x^j y^k f(\boldsymbol{e}_k, \boldsymbol{e}_j) \\ \implies f\left(x^j \boldsymbol{e}_j, y^k \boldsymbol{e}_k\right) &= f\left(y^k \boldsymbol{e}_k, x^j \boldsymbol{e}_j\right) \\ \implies f(\boldsymbol{x}, \boldsymbol{y}) &= f(\boldsymbol{y}, \boldsymbol{x}). \end{aligned}$$

□

**Theorem 1.** *The set of rank-2 tensors forms a vector space of dimension $n^2$ over $\mathbb{R}$.*

*Proof.* (Problem #3) □

In a similar manner to how we constructed 2-tensors, one can consider *multilinear forms* $f : V \times V \times V \to \mathbb{R}$, $f : V \times V \times V \times V \to \mathbb{R}$, etc. to construct tensors of rank 3, 4, etc. with coordinates $t_{jkl}$, $t_{jklm}$, etc. Having defined tensors in this manner, let us consider some commonly encountered tensors.

**Example 1.** ***The Levi-Civita tensor.*** Consider $\mathbb{R}^3$ with its Cartesian basis $\{e_1, e_2, e_3\}$. The *Levi-Civita tensor* (or *completely antisymmetric tensor*) is the rank-3 tensor $\varepsilon : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\varepsilon(\boldsymbol{e}_j, \boldsymbol{e}_k, \boldsymbol{e}_l) =: \varepsilon_{jkl} = \begin{cases} +1 & \text{if } (jkl) \text{ is an even permutation of } (123) \\ -1 & \text{if } (jkl) \text{ is an odd permutation of } (123) \\ 0 & \text{if } (jkl) \text{ is not a permutation of } (123). \end{cases}$$

One example of its use is in representing the cross product $x \times y$ in Einstein notation:

$$(\boldsymbol{x} \times \boldsymbol{y})_j = \varepsilon_{jkl} x^k y^l.$$

**Example 2.** ***The Euclidean Kronecker delta.*** Consider $\mathbb{R}^n$ with its Cartesian basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. The *Euclidean Kronecker delta* is the rank-2 tensor $\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where

$$\delta(\boldsymbol{e}_j, \boldsymbol{e}_k) =: \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\delta_{jk}$ has the values 0 and 1 in the particular case of the Cartesian basis, but generally this is not so. This is because the Kronecker delta is typically defined in terms of the mixed tensor, $\delta^j_k$, which we discuss in the next section.

## 1.3 Dual spaces

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$, and let $f$ be a linear form thereon. Let $\boldsymbol{x} \in V$, and expand $\boldsymbol{x}$ in a basis: $\boldsymbol{x} = x^j \boldsymbol{e}_j$. Now consider

$$\begin{aligned} f(\boldsymbol{x}) = f\left(x^1 \boldsymbol{e}_1 + \cdots + x^n \boldsymbol{e}_n\right) &= f(\boldsymbol{e}_1) x^1 + \cdots + f(\boldsymbol{e}_n) x^n \\ &=: u_1 x^1 + \cdots + u_n x^n = u_j x^j \end{aligned}$$

where $u_j := f(\boldsymbol{e}_j) \in \mathbb{R}$. Every linear form on $V$ can be written in this way; the scalars $u_j$ uniquely determine the form $f$.[3] Furthermore, the set of all $\boldsymbol{u} := (u_1, \dots u_n)$, and thus the set of linear forms $f$, constitutes a vector space, denoted $V^*$. Since $V^*$ is of dimension $n$, it is isomorphic to $\mathbb{R}^n$, and by extension, to $V$.

**Definition 1.** ***Dual spaces.***
(a) The space $V^*$ of linear forms on $V$ is called the space *dual* to $V$.
(b) The elements of $V^*$ are called *co-vectors.*[a] They are one-to-one correspondent to the vector elements of $V$.

[a] Co-vectors are also called *covariant vectors*, in which case vectors are called *contravariant vectors*.

Since the co-vectors are defined via linear forms, and rank-$n$ tensors are defined by $n$-linear forms,[4] we can consider co-vectors as tensors of rank 1.

**Definition 2.** ***Natural pairing.*** The scalar $f(\boldsymbol{x}) \in \mathbb{R}$ is called the *natural pairing* or *dual pairing*[a] of the co-vector $\boldsymbol{u}$ (corresponding to $f$) and the vector $\boldsymbol{x}$. We write

$$\langle \boldsymbol{u}, \boldsymbol{x} \rangle := f(\boldsymbol{x}) = u_j x^j.$$

[a] According to Dr. Belitz, this is called the *scalar product* and denoted $\boldsymbol{u} \cdot \boldsymbol{x}$, though I have been unable to verify this.

If $\{\boldsymbol{e}_j\}$ is a basis of $V$, there exists a canonical *dual basis* or *co-basis* $\{\boldsymbol{e}^j\}$ of $V^*$,[5] defined by

$$\langle \boldsymbol{e}^j, \boldsymbol{e}_k \rangle = (\boldsymbol{e}^j)_l (\boldsymbol{e}_k)^l = \delta^j_k,$$

where

$$\boxed{\delta^j_k := \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}}$$

is called the *Kronecker delta*. The basis $\{\boldsymbol{e}_j\}$ and co-basis $\{\boldsymbol{e}^j\}$ are said to be *biorthogonal*.[6] Any element $u \in V^*$ can be expanded in terms of the dual basis as

$$\boldsymbol{u} = u_j \boldsymbol{e}^j.$$

**Definition 3.** ***Contra-/co-variant and mixed tensors.***
(a) Bilinear forms $f : V^* \times V^* \to \mathbb{R}$ acting on the co-basis define *contravariant tensors* of rank 2,

$$f(\boldsymbol{e}^j, \boldsymbol{e}^k) = t^{jk},$$

and analogously for higher rank tensors.[a] The tensors of Example 2 are then called *covariant tensors*.
(b) Multilinear forms acting on mixtures of basis and co-basis vectors define *mixed tensors*. For example, $f : V^* \times V \times V^* \to \mathbb{R}$ defines $t^{jl}_{k} = f(\boldsymbol{e}^j, \boldsymbol{e}_k, \boldsymbol{e}^l)$.[b]

[a] In this manner, vectors can be considered contravariant tensors of rank 1.
[b] The Kronecker delta, $\delta^j_k$, is thus a mixed tensor of rank 2.

**Definition 4.** ***Tensor product.*** The contravariant tensor whose components are given by the product of the components of two contravariant vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is called the *tensor product* of $\boldsymbol{x}$ and $\boldsymbol{y}$, denoted by

$$\boxed{t = \boldsymbol{x} \otimes \boldsymbol{y}, \quad t^{jk} = x^j y^k}$$

[3] The proof of this is analogous to that of Proposition 1.
[4] As per § 1.2
[5] It can be proven that, for finite dimensional $V$, the co-basis is a basis of $V^*$.
[6] If $\{\boldsymbol{e}_j\}$ is the Cartesian basis, it can be proven that $\boldsymbol{e}_j = \boldsymbol{e}^j \, \forall j \in [n]$

Analogously, $t_{jk} = x_j y_k$, $t_j{}^k = x_j y^k$, and $t^j{}_k = x^j y_k$.

# 2 Minkowski space

## 2.1 The metric tensor

**Definition 1.** ***Metric tensor.*** Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ with basis $\{\boldsymbol{e}_j\}$, and let $g : V \times V \to \mathbb{R}$ be a symmetric bilinear form.[a] Then $g$ defines a symmetric 2-tensor:

$$\boxed{g_{jk} = g\left(\boldsymbol{e}_j, \boldsymbol{e}_k\right) = g_{kj}}$$

Let $g$ have an inverse $g^{-1}$, corresponding to a tensor $g^{jk}$, in the sense

$$\boxed{g_{jk} g^{kl} = \delta^l_j}$$

Then we call the scalar

$$\boxed{g\left(\boldsymbol{x}, \boldsymbol{y}\right) = x^j g_{jk} y^k}$$

the *generalized scalar product of $x$ and $y$*, with $g_{jk}$ called the *metric tensor*, denoted

$$\boxed{g\left(\boldsymbol{x}, \boldsymbol{y}\right) =: \boldsymbol{x} \cdot \boldsymbol{y} =: \boldsymbol{xy}}$$

[a] That is, $g\left(\boldsymbol{x}, \boldsymbol{y}\right) = g\left(\boldsymbol{y}, \boldsymbol{x}\right)\ \forall \boldsymbol{x}, \boldsymbol{y} \in V$.

Since $V$ is isomorphic to $\mathbb{R}^n$, we can consider $\mathbb{R}^n$ in what follows.

**Definition 2.** ***Co-basis.*** Consider $\mathbb{R}^n$ endowed with a metric tensor, $g$, and let $\{\boldsymbol{e}_j\}$ be a basis. We define an *adjoint basis* or *co-basis* $\left\{\boldsymbol{e}^j\right\}$ *by*

$$\boxed{\boldsymbol{e}^j := g^{jk} \boldsymbol{e}_k}$$

It readily follows that[a]

$$\boxed{\boldsymbol{e}_j = g_{jk} \boldsymbol{e}^k}$$

[a] $\boldsymbol{e}_j = \delta^l_j \boldsymbol{e}_l = g_{jk} g^{kl} \boldsymbol{e}_l = g_{jk} \boldsymbol{e}^k$

**Proposition 1.** *The contravariant and covariant components of a vector are related by*

$$\boxed{x_j = g_{jk} x^k, \quad x^j = g^{jk} x_k}$$

*Proof.* $\boldsymbol{x} = x^j \boldsymbol{e}_j = x^j g_{jk} \boldsymbol{e}^k = x_k \boldsymbol{e}^k$, since $x_j$ are defined to be the components of $\boldsymbol{x}$ in basis $\left\{\boldsymbol{e}^j\right\}$. But $g$ is symmetric:
$g_{jk} = g_{kj} \implies x_k = g_{kj} x^j$. Therefore, $x^j = \delta^k_j x^k = g_{kl} g^{lj} x^k = x_l g^{lj}$. □

*Remark* 1. It can be proven[7] that the metric tensor, operating on a general $n+1$ rank tensor, has the effect that it lowers or raises the index being summed over:

$$g_{jk} t^{k l_1 \cdots l_n} = t_j{}^{l_1 \cdots l_n}.$$

[7] We proved it in class lol

*Remark* 2. Note that $\delta_{jk} = g_{jl}\delta^k_l = g_{jk}$, which in general is *not* equal to $\delta^j_k$. However, $g^l_j = g_{jk}g^{kl} = \delta^l_j$ is always true. Only in Euclidean space is $\delta_{jk} = \delta^j_k$.

## 2.2 Basis transformations

**Definition 1.** ***Matrices.***
(a) An $n \times n$ array of real numbers $D^j{}_k$ (corresponding to the $j^{th}$ row and $k^{th}$ column) we call an $n \times n$ *matrix* $D$ with *elements* $D^j{}_k$.
(b) A matrix $D$ is *invertible* if a matrix $D^{-1}$ exists such that

$$\boxed{D^j{}_k(D^{-1})^k{}_l = (D^{-1})^j{}_k D^k{}_l = \delta^j_l}$$

or, $DD^{-1} = \mathbb{1}_n$ with $\mathbb{1}_n$ the $n \times n$ unit matrix with $(\mathbb{1}_n)^j{}_k = \delta^j_k$.
(c) The matrix $D^T$ with elements

$$\boxed{(D^T)^j{}_k = D_k{}^j}$$

is called the *transpose* of $D$.

**Proposition 1.** *The transpose of a product is the product of the transposes, in reverse order:*

$$\boxed{(AB)^T = B^T A^T}$$

*Proof.* $((AB)^T)^j{}_k = (AB)_k{}^j = A_k{}^l B_l{}^j = (A^T)^l{}_k (B^T)^j{}_l = (B^T)^j{}_l (A^T)^l{}_k = (B^T A^T)^j{}_k$. □

**Proposition 2.** *The inverse of a transpose is the transpose of the inverse:*

$$\boxed{\left(D^{-1}\right)^T = \left(D^T\right)^{-1}}$$

*Proof.* $D^T \left(D^{-1}\right)^T = \left(D^{-1}D\right)^T = (\mathbb{1}_n)^T = \mathbb{1}_n$. □

**Definition 2.** ***Basis transformation.*** Consider $\mathbb{R}^n$ with a metric tensor $g$; let $\{\boldsymbol{e}_j\}$ be a basis and let $D$ be an invertible matrix. Then we define a new basis[a] $\{\tilde{\boldsymbol{e}}_j\}$ by the *basis transformation*

$$\boxed{\tilde{\boldsymbol{e}}_j = \boldsymbol{e}_k (D^{-1})^k{}_j},$$

whose inverse transformation yields[b]

$$\boxed{\boldsymbol{e}_j = \tilde{\boldsymbol{e}}_k D^k{}_j}$$

[a] For proof that it is indeed a basis, see Problem 5.
[b] $\boldsymbol{e}_j = \boldsymbol{e}_k \delta^k_j = \boldsymbol{e}_k (D^{-1})^k{}_l D^l{}_j = \tilde{\boldsymbol{e}}_l D^l{}_j$

**Proposition 3.** *Let $\boldsymbol{x} \in \mathbb{R}^n$ be a vector whose contravariant coordinates with respect to a basis $\{\boldsymbol{e}_j\}$ are $x^j$. Then its coordinates with respect to the basis $\{\tilde{\boldsymbol{e}}_j\}$, denoted $\tilde{x}^j$ and called a* coordinate transformation, *are*

$$\boxed{\tilde{x}^j = D^j{}_k x^k} \quad \text{or} \quad \boxed{\tilde{\boldsymbol{x}} = D\boldsymbol{x}}$$

*with inverse transformation*

$$\boxed{x^j = (D^{-1})^j{}_k \tilde{x}^k =} \quad \text{or} \quad \boxed{\boldsymbol{x} = D^{-1}\tilde{\boldsymbol{x}}}$$

*Proof.* $\boldsymbol{x} = x^j \boldsymbol{e}_j = x^j \tilde{\boldsymbol{e}}_k D^k{}_j = x^j D^k{}_j \tilde{\boldsymbol{e}}_k = \tilde{x}^k \tilde{\boldsymbol{e}}_k \implies \tilde{x}^k = x^j D^k{}_j$ □

**Proposition 4.** *Let $g_{jk} = \boldsymbol{e}_j \cdot \boldsymbol{e}_k$ be the metric in the basis $\{\boldsymbol{e}_j\}$, and let $D^{-1}$ be a basis transformation such that $\tilde{\boldsymbol{e}}_j = \boldsymbol{e}_k (D^{-1})^k{}_j$. Then the metric $\tilde{g}$ corresponding to $g$ expressed in the transformed basis $\{\tilde{\boldsymbol{e}}_j\}$, defined by coordinates*

$$\boxed{\tilde{g} := g\left(\tilde{\boldsymbol{e}}_j, \tilde{\boldsymbol{e}}_k\right)}$$

*is given by*

$$\boxed{\tilde{g}_{jk} = ((D^{-1})^T)_j{}^m g_{ml} (D^{-1})^l{}_k} \quad \text{or} \quad \boxed{\tilde{g} = \left(D^{-1}\right)^T g D^{-1}} \quad \text{or} \quad \boxed{g = D^T \tilde{g} D}$$

*Proof.* Problem 6 □

**Corollary 1.** *The covariant coordinates transform according to*

$$\boxed{\tilde{x}_j = (D^{-1})^k{}_j x_k}$$

*with inverse transformation*

$$\boxed{x_j = D^k{}_j \tilde{x}_k}$$

*Proof.* $\tilde{x}_j = \tilde{g}_{jk} \tilde{x}^k = \tilde{g}_{jk} D^k{}_l x^l = \tilde{g}_{jk} D^k{}_l g^{lm} x_m = (\tilde{g} D g)_j{}^m x_m$
$= (\left(D^{-1}\right)^T g D^{-1} D g)_j{}^m x_m \overset{g^2 = \mathbb{1}}{=} (\left(D^{-1}\right)^T)_j{}^m x_m = (D^{-1})^m{}_j x_m$ □

## 2.3 Normal coordinate systems

**Lemma 1.** *For every symmetric $n \times n$ matrix $M^j{}_k = M^k{}_j$ that has an inverse, there exists a transformation $D$ such that*[a]

$$\tilde{M}^j{}_k = (D^T M D)^j{}_k = m_{(j)} \delta^j_k.$$

*That is to say, there exists a transformation that diagonalizes $M$.*

[a]When a sub- or superscript is in parentheses, no summation is implied.

*Proof.* This is called the *spectral decomposition theorem*, and is proven elsewhere. □

**Corollary 1.** *Let $g_{jk}$ be a metric on $\mathbb{R}^n$. There exists a coordinate transformation $D$ such that*[a]

$$\boxed{\tilde{g}_{jk} = \lambda_{(j)}\delta_{jk}}$$

[a]Here, for whatever reason, the Kronecker delta is the Euclidean one.

*Proof.* $g$ can be considered a real symmetric matrix; by Lemma 1, it can be diagonalized in this form. □

**Theorem 1.** *There exists a coordinate transformation $D$ that diagonalizes $g$ such that*

$$\boxed{\tilde{g} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}}$$

*with $m$ elements of $+1$ and $n-m$ elements of $-1$, where $0 \leq m \leq n$.*

*Proof.* From Corollary 1, we can write ... □

**Definition 1.** ***Normed coordinate systems.*** Basis sets in which the metric has the form of Theorem 1 are called *normed coordinate systems*. The number $m$ is characteristic of the space; this is sometimes called *Sylvester's Rigidity Theorem.*

**Example 1.** ***Normed Euclidean space.*** Let $m = n$. Then

$$g = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

and we see

$$g_{jk} = \delta^j_k.$$

$\mathbb{R}^n$ endowed with this metric is called *$n-$dim Euclidean Space, $E_n$*. The normal coordinate systems are called *Cartesian*. In the space $E_n$, we have $x_j = g_{jk}x^k = \delta^j_k x^k = x^j$. In this case, positive semi-definiteness holds, and so also the Pythagorean Theorem.

**Example 2.** ***Normed Minkowski space.*** Let $m = 1$; $(n \geq 2)$. Then

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

$\mathbb{R}^n$ endowed with this metric is called *Minkowski space*, $M_n$. The normal coordinate systems are called *inertial frames.* In the space $M_n$, we have $x_1 = x^1$ and $x_j = -x^j$ for $j = 2, \ldots, n$. In physics we label $\boldsymbol{x} = \left(x^0, x^1, x^2, x^3\right)$ with $x^0 = x_0 = ct$ where $t$ is called *time* and $\left(x^1, x^2, x^3\right)$ is called space. $c$ is a characteristic velocity, namely the speed of light in vacuum.

## 2.4 Normal coordinate transformations

**Definition 1.** ***Normal coordinate transformation.*** A *normal coordinate transformation* is one that transforms a normal coordinate system into another normal coordinate system. That is,

$$\boxed{g = \left(D^{-1}\right)^T g D^{-1}},$$

from which it follows

$$\boxed{g = D^T g D}.$$

**Example 1.** For $E_n$, these transformations are called *orthogonal* and are a subset of *unitary transformations:*

$$g = \mathbb{1}_n = \left(D^{-1}\right)^T \mathbb{1}_n D^{-1} = \left(D^{-1}\right)^T D^{-1}$$
$$\implies D^T = D^{-1}$$

**Example 2.** For $M_4$, these transformations are called *Lorentz transformations.*

**Lemma 1.**
*(i) If $D$ is a normal transformation, then so is $D^{-1}$*
*(ii) If $D_1$, $D_2$ are normal transformations, then so is the successive transformation $D_1 D_2$.*

*Proof.* Problem 7 □

**Theorem 1.** *The set of normal transformations forms a group (not necessarily abelian) under matrix multiplication.*

*Proof.* Problem 7 □

**Example 3.** In $E_n$, the group of normal transformations is called the *orthogonal group* $O\left(n\right)$.

**Example 4.** In $M_n$, the group of normal transformations is called the *pseudo-orthogonal group* $O\left(1, n-1\right)$.

**Proposition 1.** *Let $D$ be a normal coordinate transformation. Then*

$$\boxed{\det D = \pm 1}$$

*Proof.* From Definition 1, $\det g = \det\left(D^T g D\right) = \det g (\det D)^2 \implies \det D = \pm 1$. □

# 3 Tensor Fields

## 3.1 The concept of a tensor field

Let $V$ be $\mathbb{R}^n$ endowed with metric tensor $g$, and let $D$ be a normal coordinate transformation (say, from coordinate system $CS$ to $\widetilde{CS}$). That is, transformed coordinates take the form $\tilde{x}^j = D^j{}_k x^k$.

**Definition 1.** ***(Pseudo-)tensor fields.*** $\forall \boldsymbol{x} \in V$, consider assigning a rank-$N$ tensor $t^{j_1 \cdots j_N}(\boldsymbol{x})$ to $\boldsymbol{x}$. The set of assigned tensors,[a] $\left\{t^{j_1 \cdots j_N}(\boldsymbol{x}); \boldsymbol{x} \in V\right\}$, is called a *tensor field* iff, under a coordinate transformation,

$$\boxed{\tilde{t}^{j_1 \cdots j_N}(\boldsymbol{x}) = D^{j_1}{}_{k_1} \cdots D^{j_N}{}_{k_N} t^{j_1 \cdots j_N}(\boldsymbol{x})}$$

and is called a *pseudo-tensor field* iff, under a coordinate transformation,

$$\boxed{\tilde{t}^{j_1 \cdots j_N}(\boldsymbol{x}) = (\det D)\, D^{j_1}{}_{k_1} \cdots D^{j_N}{}_{k_N} t^{j_1 \cdots j_N}(\boldsymbol{x})}$$

[a] A tensor field $t$ can be considered a tensor-valued function with domain $V$. That is, $t : V \to V^N$.

**Example 1.** Is the Levi-Civita tensor a tensor or pseudo-tensor? Recall that by Example 1, the Levi-Civita is independent of $\boldsymbol{x}$; that is,[a]

$$\tilde{\varepsilon}^{jkl} := \varepsilon\left(\tilde{\boldsymbol{e}}^j, \tilde{\boldsymbol{e}}^k, \tilde{\boldsymbol{e}}^l\right) = \varepsilon\left(\boldsymbol{e}^j, \boldsymbol{e}^k, \boldsymbol{e}^l\right) = \varepsilon^{jkl}.$$

Let $D$ be a normal coordinate transformation. Then[b]

$$\begin{aligned}
D^j{}_\alpha D^k{}_\beta D^l{}_\gamma \varepsilon^{\alpha\beta\gamma} &= \sum_{\pi \in S_3} \operatorname{sgn}(\pi)\, D^j{}_{\pi(1)} D^k{}_{\pi(2)} D^l{}_{\pi(3)} \\
&= \begin{vmatrix} D^j{}_1 & D^j{}_2 & D^j{}_3 \\ D^k{}_1 & D^k{}_2 & D^k{}_3 \\ D^l{}_1 & D^l{}_2 & D^l{}_3 \end{vmatrix} \\
&= \operatorname{sgn}\begin{pmatrix} j & k & l \\ 1 & 2 & 3 \end{pmatrix} \det D = \varepsilon^{jkl} \det D \\
&= \varepsilon^{jkl} \frac{1}{\det D} = \tilde{\varepsilon}^{jkl} \frac{1}{\det D}.
\end{aligned}$$

As per Definition 1, the Levi-Civita tensor constitutes a pseudo-tensor field:

$$\boxed{\tilde{\varepsilon}^{jkl} = (\det D)\, D^j{}_\alpha D^k{}_\beta D^l{}_\gamma \varepsilon^{\alpha\beta\gamma}}$$

[a] This is the only justification I could come up with for this equation. I am not sure if the middle two parts of this equation are the reason you can say this.

[b] A few remarks on the notation used here:

- $S_3$ denotes the "symmetric group on 3 letters", and represents the set of possible permutations of the set $\{1, 2, 3\}$. Thus, $\pi \in S_3$ means $\pi$ is some permutation of the numbers $\{1, 2, 3\}$ such as 312, and $\sum_{\pi \in S_3} (\cdots)$ represents a sum over all possible permutations of $\{1, 2, 3\}$.
- $\operatorname{sgn}(\pi)$ represents the "sign" of the permutation; if it is an even permutation, $\operatorname{sgn}(\pi) = 1$, and if an odd permutation, $\operatorname{sgn}(\pi) = -1$.

- $\pi(1)$ is the first number in the permutation, $\pi(2)$ is the second, etc. That is, if the permutation is 312, $\pi(1) = 3$, $\pi(2) = 1$, etc.
- We can represent a permutation, say 312, with the notation $\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. The order of the columns doesn't matter: $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$. However, the bottom row always represents $\pi(1)$, $\pi(2)$, etc.

## 3.2 Gradient, curl, divergence

Let $f(\boldsymbol{x})$ be a scalar-valued function $f : V \to \mathbb{R}$ (that is, a scalar field).

*Claim* 1. Let $D$ be a coordinate transformation. Then

$$\boxed{\left(D^{-1}\right)^j{}_k = \frac{\partial x^j}{\partial \tilde{x}^k}}$$

*Proof.* Take the partial derivative of $x^j = \left(D^{-1}\right)^j{}_k \tilde{x}^k$ with respect to $\tilde{x}^k$. □

**Definition 1.** ***Gradient.*** The *gradient* of $f$, denoted $(\nabla f)(\boldsymbol{x})$ or $(\text{grad } f)(\boldsymbol{x})$, is the vector field defined by components[a]

$$\boxed{(\nabla f)_j(\boldsymbol{x}) := \frac{\partial}{\partial x^j} f(\boldsymbol{x})}$$

which is often also written[b]

$$\boxed{\partial_j f(\boldsymbol{x}) := \frac{\partial}{\partial x^j} f(\boldsymbol{x})}.$$

Analogously,[c]

$$\boxed{\partial^j f(\boldsymbol{x}) := \frac{\partial}{\partial x_j} f(\boldsymbol{x})}.$$

[a]Note that when we begin discussing Minkowski space, the $\nabla$ symbol will be reserved for Euclidean vectors.
[b]A subscript is used because, as we will prove, the gradient transforms covariantly.
[c]The superscript reflects the fact that this derivative transforms contravariantly.

**Proposition 1.** *The gradient of a scalar field transforms as a covariant vector:*

$$\boxed{\tilde{\partial}_j \tilde{f}(\tilde{\boldsymbol{x}}) = \left(D^{-1}\right)^k{}_j \partial_k f(\boldsymbol{x})}$$

*Proof.* Let $D$ be a coordinate transformation. Then[a]

$$\begin{aligned}
\tilde{\partial}_j \tilde{f}(\tilde{\boldsymbol{x}}) &= \frac{\partial}{\partial \tilde{x}^j} \tilde{f}(\tilde{\boldsymbol{x}}) = \frac{\partial}{\partial \tilde{x}^j} f(\boldsymbol{x}) \\
&= \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial}{\partial x^k} f(\boldsymbol{x}) \\
&= \left(D^{-1}\right)^k{}_j \partial_k f(\boldsymbol{x}).
\end{aligned}$$

□

[a] Since $f$ is a scalar-valued function, $\tilde{f}(\tilde{\boldsymbol{x}}) = f(\boldsymbol{x})$. The second line follows from the chain rule.

**Definition 2.** ***Curl.*** The *curl* of a vector field $\boldsymbol{v}(\boldsymbol{x})$, denoted $(\nabla \times \boldsymbol{v})(\boldsymbol{x})$ or $(\operatorname{curl} \boldsymbol{v})(\boldsymbol{x})$, is the vector field whose $j^{\text{th}}$ component is[a]

$$\boxed{(\nabla \times \boldsymbol{v})^j(\boldsymbol{x}) := \varepsilon^{jkl}\partial_k v_l(\boldsymbol{x})}$$

[a] The superscript reflects the fact that the curl transforms as a pseudovector.

**Proposition 2.** *The curl of a vector field transforms as a pseudovector:*

$$\boxed{\widetilde{(\nabla \times \boldsymbol{v})}^j(\tilde{\boldsymbol{x}}) = (\det D)\, D^j{}_k\, (\nabla \times \boldsymbol{v})^k(\boldsymbol{x})}$$

*Proof.* Let $D$ be a coordinate transformation. By Proposition 1 and Corollary 1 from § 2.2,

$$\begin{aligned}
\widetilde{(\nabla \times \boldsymbol{v})}^j(\tilde{\boldsymbol{x}}) &= \tilde{\varepsilon}^{jkl}\tilde{\partial}_k\tilde{v}_l(\tilde{\boldsymbol{x}}) \\
&= \tilde{\varepsilon}^{jkl}\left(D^{-1}\right)^m{}_k\partial_m\left(D^{-1}\right)^\alpha{}_l v_\alpha(\boldsymbol{x}) \\
&= \delta^j_\beta\tilde{\varepsilon}^{\beta kl}\left(D^{-1}\right)^m{}_k\left(D^{-1}\right)^\alpha{}_l\partial_m v_\alpha(\boldsymbol{x}) \\
&= D^j{}_\gamma\underbrace{\left(D^{-1}\right)^\gamma{}_\beta\tilde{\varepsilon}^{\beta kl}\left(D^{-1}\right)^m{}_k\left(D^{-1}\right)^\alpha{}_l}_{(\det D)\varepsilon^{\gamma m\alpha}}\partial_m v_\alpha(\boldsymbol{x}) \\
&= (\det D)\, D^j{}_\gamma\varepsilon^{\gamma m\alpha}\partial_m v_\alpha(\boldsymbol{x}) \\
&= (\det D)\, D^j{}_\gamma(\nabla \times \boldsymbol{v})^\gamma(\boldsymbol{x})
\end{aligned}$$

□

**Definition 3.** ***Divergence.*** The *divergence* of a vector field $\boldsymbol{v}(\boldsymbol{x})$, denoted $(\nabla \cdot \boldsymbol{v})(\boldsymbol{x})$ or $(\operatorname{div} \boldsymbol{v})(\boldsymbol{x})$, is the scalar field defined by

$$\boxed{(\nabla \cdot \boldsymbol{v})(\boldsymbol{x}) := \partial_j v^j(\boldsymbol{x})}$$

**Proposition 3.** *The divergence of a vector field transforms as a scalar:*

$$\boxed{\widetilde{(\nabla \cdot \boldsymbol{v})}(\tilde{\boldsymbol{x}}) = (\nabla \cdot \boldsymbol{v})(\boldsymbol{x})}$$

*Proof.* By Proposition 1 and Proposition 3 from § 2.2,

$$\begin{aligned}
\widetilde{(\nabla \cdot \boldsymbol{v})}(\tilde{\boldsymbol{x}}) &= \tilde{\partial}_j\tilde{v}^j(\tilde{\boldsymbol{x}}) \\
&= \left(D^{-1}\right)^l{}_j\partial_l D^j{}_k v^k(\boldsymbol{x}) \\
&= \left(D^{-1}\right)^l{}_j D^j{}_k\partial_l v^k(\boldsymbol{x}) \\
&= \delta^l_k\partial_l v^k(\boldsymbol{x}) = \partial_k v^k(\boldsymbol{x}) \\
&= (\nabla \cdot \boldsymbol{v})(\boldsymbol{x})
\end{aligned}$$

□

## 3.3 Tensor products and traces

We can generalize the concepts of the tensor product defined in Ch. 1 and the trace of a matrix.

**Definition 1.** ***(General) tensor product.*** Let $s$, $t$ be tensors of ranks $N$ and $M$, respectively. The *tensor product* of $s$ and $t$, denoted $s \otimes t$, is the rank $N+M$ tensor defined by coordinates

$$\boxed{(s \otimes t)^{j_1 \cdots j_{N+M}} = s^{j_1 \cdots j_N} t^{j_{N+1} \cdots j_{N+M}}}$$

**Proposition 1.** *The tensor product of two tensors or pseudotensors is tensor, while the tensor product of a tensor with a pseudotensor is a pseudotensor.*

*Proof.* Easy (apparently) □

**Definition 2.** ***Contraction.*** Let $t$ be a tensor or pseudotensor of rank $N+2$. We define the $(1,2)-$*trace* or $(1,2)-$*contraction* of $t$ as the rank $N$ tensor or pseudotensor $u$ with components

$$\boxed{u^{l_1 \cdots l_N} := g_{jk} t^{jkl_1 \cdots l_N} = t^{j}{}_{j}{}^{l_1 \cdots l_N} = t_j{}^{jl_1 \cdots l_N}}.$$

Note that the 1st and 2nd indices were summed over; in general the $(j,k)-$*contraction* will instead sum over the $j^{\text{th}}$ and $k^{\text{th}}$ indices, respectively.

**Example 1.** The curl of a vector field can be considered a $(2,4)-$ and $(3,5)-$contraction of the rank 5 pseudotensor $\varepsilon^{jkl} \partial^m v^\alpha (\boldsymbol{x})$.

## 3.4 Minkowski tensors

Consider $M_4$; that is, $\mathbb{R}^4$ endowed with the Minkowski metric tensor $g = (+,-,-,-)$.

Let $A^\mu \in M_4$. We adopt the following conventions:

1. We will often refer to the entire vector as $A^\mu = \left(A^0, A^1, A^2, A^3\right) =: \left(A^0, \boldsymbol{A}\right)$.[8]
2. In sums, lowercase Greek indices run over all four indices: $\mu = 0,1,2,3$.
3. Latin indices run over the three Euclidean components: $j = 1,2,3$.

In this notation, $A_\mu = g_{\mu\nu} A^\nu = \begin{cases} A^0 & \mu = 0 \\ -A^j & \mu = 1,2,3 \end{cases}$. Furthermore, $\boldsymbol{A} := \left(A^1, A^2, A^3\right)$ can be considered a Euclidean vector in the subspace of $M_4$ spanned by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

Now consider a rank 2 tensor $F$. Analogous to the above conventions, we can write $F$ in an array as

$$\begin{aligned} F^{\mu\nu} &= \left(\begin{array}{c:ccc} F^{00} & F^{01} & F^{02} & F^{03} \\ \hdashline F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{array}\right) \\ &= \left(\begin{array}{c:c} F^{00} & F^{0j} \\ \hdashline F^{j0} & F^{jk} \end{array}\right) \end{aligned}$$

[8] Note that we also label the first index with a 0 instead of a 1.

where $F^{0j}$ and $F^{j0}$ can be considered vectors in the Euclidean subspace, and $F^{jk}$ can be considered a Euclidean 2-tensor.[9]

**Definition 1.** ***Symmetric tensors.*** $F^{\mu\nu}$ is called a *symmetric tensor* iff

$$\boxed{F^{\mu\nu} = F^{\nu\mu}},$$

from which it follows that $F^{0j} = F^{j0}$.

**Definition 2.** ***Antisymmetric tensors.*** $F^{\mu\nu}$ is called an *antisymmetric tensor* iff

$$\boxed{F^{\mu\nu} = -F^{\nu\mu}},$$

from which it follows that $F^{jk}$ is antisymmetric, $F^{0j} = -F^{j0}$ and $F^{\mu\mu} = 0$.

**Lemma 1.** *Antisymmetric Euclidean 2-tensors are isomorphic to Euclidean pseudovectors.*

*Proof.*

$$\begin{aligned} t^{jk} = -t^{kj} \quad &\Longrightarrow \quad t = \begin{pmatrix} 0 & v^3 & -v^2 \\ -v^3 & 0 & v^1 \\ v^2 & -v^1 & 0 \end{pmatrix} \\ &\Longrightarrow \quad t^{jk} = \varepsilon^{jkl} v_l \end{aligned}$$

Since $t$ is a tensor and $\varepsilon$ is a pseudotensor, $v$ is a pseudovector.[a] □

[a]For some reason, Belitz writes the components of $\boldsymbol{v}$ with subscripts instead of superscripts. I think they should be superscripts, though.

**Proposition 1.** *Any antisymmetric 2-tensor in Minkowski space can be written*

$$\begin{aligned} F^{\mu\nu} &= \left(\begin{array}{c|c} 0 & a^j \\ \hline -a^j & t^{jk} \end{array}\right) \\ &:= \left(\begin{array}{c|ccc} 0 & a^1 & a^2 & a^3 \\ \hline -a^1 & 0 & v^3 & -v^2 \\ -a^2 & -v^3 & 0 & v^1 \\ -a^3 & v^2 & -v^1 & 0 \end{array}\right) \end{aligned}$$

*where* $\boldsymbol{a}$ *is a Euclidean vector and* $\boldsymbol{v} := \left(v^1, v^2, v^3\right)$ *is a Euclidean pseudovector.*

*Proof.* See Definition 2 and Lemma 1. □

[9]I use a different notation that Belitz; in his notation, $F^{0j} = \boldsymbol{F}_h$ and $F^{j0} = \boldsymbol{F}_v$.

**Example 1.**

$$
\begin{aligned}
F^{\mu}{}_{\nu} = F^{\mu\kappa} g_{\kappa\nu} &= \left(\begin{array}{c:c} 0 & a^j \\ \hdashline -a^j & t^{jk} \end{array}\right) \left(\begin{array}{c:c} 1 & \\ \hdashline & -\mathbb{1}_3 \end{array}\right) \\
&= \left(\begin{array}{c:c} 0 & -a^j \\ \hdashline -a^j & -t^{jk} \end{array}\right)
\end{aligned}
$$

Follow an analogous procedure to compute

$$
F_{\mu\nu} = \left(\begin{array}{c:c} 0 & -a^j \\ \hdashline a^j & t^{jk} \end{array}\right)
$$

$$
F_{\mu}{}^{\nu} = \left(\begin{array}{c:c} 0 & a^j \\ \hdashline a^j & -t^{jk} \end{array}\right)
$$

**Example 2.** $F_{\mu\nu}F^{\mu\nu} = 2\left(\boldsymbol{v}^2 - \boldsymbol{a}^2\right)$. This is just a Minkowski scalar!

# Chapter 2

# Maxwell's Equations

## 1 The variational principle of classical electrodynamics

### 1.1 The Maxwellian action

**Axiom 1.** Space and time are described by a four-dimensional Minkowski space with elements

$$\boxed{x^\mu = (ct_x, \boldsymbol{x})},$$

where $t_x$ is called *time*, $\boldsymbol{x}$ is the position in space, and $c$ is a characteristic velocity.

*Remark* 1. We adopt the conventions outlined in Ch. 1, § 3.4.

**Axiom 2.** Empty space ("vacuum") supports a Minkowski vector field

$$A^\mu(x)$$

called the *electromagnetic 4-vector potential.*

**Definition 1. *Electromagnetic field tensor.*** The antisymmetric 2-tensor field constructed from the 4-gradients of the electromagnetic 4-vector potential via

$$\boxed{F^{\mu\nu}(x) := \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)}$$

is called the *electromagnetic field tensor.*

*Remark* 2. By Ch. 1, § 3.4, $F^{\mu\nu}(x)$ can be represented in terms of a Euclidean vector field and a Euclidean pseudovector field.

**Axiom 3(a).** The physical field configurations in vacuum are those that minimize the action

$$\boxed{S_{\text{vac}} := -\frac{1}{16\pi}\int d^4x\, F_{\mu\nu}(x)\, F^{\mu\nu}(x)},$$

where $d^4x := cdt\, d\boldsymbol{x}$ .

*Remark* 3. The coefficient $\frac{1}{16\pi}$ is dependent on the unit convention used. In this class we use *CGS*.

*Remark* 4. Classical electrodynamics is governed by a principle of least action, as is classical mechanics. However, in electrodynamics we need to find field configurations $A^\mu(x)$ that minimize the action; in mechanics, we only had to find paths $\boldsymbol{x}(t)$.

*Remark* 5. As per Example 2, $F^{\mu\nu}F_{\mu\nu}$ is a (Minkowski) scalar; therefore the theory is invariant under Lorentz transformations (but *not* Galilean).

**Axiom 3(b).** Matter is characterized (in part) by an $M_4$ vector $J^{\mu}(x)$ that couples to $A^{\mu}(x)$ by the action

$$\boxed{S_{\text{interaction}} := -\frac{1}{c}\int d^4x\, J^{\mu}(x)\, A_{\mu}(x)}.$$

The field plus its interaction with a *given* $J^{\mu}(x)$ is described by the action

$$\boxed{S = S_{\text{vac}} + S_{\text{interaction}}}.$$

*Remark* 6. $J^{\mu}(x)$ is called the *4-current*.

*Remark* 7. $J^{\mu}(x)$ is "god-given". We do not include the feedback from the field on the matter. One needs another action term to account for this.

**Definition 2.** ***Dual field tensor.*** The *dual*[a] *field tensor*, denoted[b] $\tilde{F}^{\mu\nu}$ is defined as

$$\boxed{\tilde{F}^{\mu\nu} := \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}},$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the *completely antisymmetric 4-tensor*.

[a]Not "dual" as in the dual vector space; this is just the conventional name for this tensor.
[b]This tilde is not implying any transformation; it is merely conventional.

**Proposition 1.**

$$\partial_{\mu}\tilde{F}^{\mu\nu}(x) = 0$$

*Proof.* Problem #12. □

## 1.2 Euler-Lagrange equations for fields

Recall that in classical mechanics, for a system with $f$ degrees of freedom, we had[1]

**Lagrangian:**

$$L = L(q_1(t), \ldots, q_f(t), \dot{q}_1(t), \ldots, \dot{q}_f(t))$$

**action:** We varied $\boldsymbol{q}(t)$ and examined $\delta S$:

$$S = \int dt\, L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$$

**extremals:**

$$\begin{aligned} 0 \overset{!}{=} \delta S &= \int dt \sum_j \left[\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j\right] \\ &= \int dt \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right]\delta q_j \\ \delta q_j \text{ arbitrary} \implies 0 &= \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \end{aligned}$$

[1]In this section and elsewhere, the symbol $\overset{!}{=}$ represents an equation we assert must be true as part of a proof (such as "we must set $\delta S = 0$ to obtain extremals" → "$\delta S \overset{!}{=} 0$").

In field theory, we follow an analogous procedure to obtain the Euler-Lagrange equations. A scalar field $\phi(x) = \phi(\boldsymbol{x}, t)$ can be considered a system with $f \to \infty$ degrees of freedom by discretizing $\phi$; to do so, we identify $\phi(\boldsymbol{x}_1, t) := q_1(t)$, $\phi(\boldsymbol{x}_2, t) := q_2(t)$, etc. Imagine dividing space into cubes with the position vector $\boldsymbol{x}_j$ pointing to the $j^{\text{th}}$ such subdivision, then taking the limit that the number of cube subdivisions goes to infinity.

We now need a "Lagrangian density" so that we can integrate over the volume elements. That is, we now have

**Lagrangian density:**

$$\mathscr{L} = \mathscr{L}\left(\phi(\boldsymbol{x}, t), \partial^\mu \phi(\boldsymbol{x}, t)\right),$$

a function that depends on spatial gradients in addition to time derivatives.

**Lagrangian:** We obtain our Lagrangian by integrating over all space, such that[2]

$$L = \int d\boldsymbol{x}\, \mathscr{L}\left(\phi(\boldsymbol{x}, t), \partial^\mu \phi(\boldsymbol{x}, t)\right).$$

**action:**

$$\begin{aligned} S = c\int dtL &= \int dx^0 \int d\boldsymbol{x}\, \mathscr{L}\left(\phi(\boldsymbol{x}, t), \partial^\mu \phi(\boldsymbol{x}, t)\right) \\ &= \int d^4x\, \mathscr{L}\left(\phi(x), \partial^\mu \phi(x)\right) \end{aligned}$$

**extremals:** As before, we require

$$\begin{aligned} 0 \overset{!}{=} \delta S &= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)\right] \\ &= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right]\delta\phi \\ \delta\phi \text{ arbitrary} \implies 0 &= \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}, \end{aligned}$$

where the second line follows from integration by parts and discarding the boundary terms.

Thus, we obtain the Euler-Lagrange equations

$$\boxed{\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathscr{L}}{\partial \phi}}. \tag{2.1}$$

*Remark* 1. These are the E-L equations for a scalar field, $\phi$. See Problem #13 for a more detailed derivation, in which the functional derivative of the action is used. For a functional $S = S[\phi(x)]$, the *functional derivative* is defined (for vectors $x, y$) as

$$\boxed{\frac{\delta S}{\delta\phi(x)} := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(S[\phi(y) + \epsilon\delta(y - x)] - S[\phi(y)]\right)}$$

*Remark* 2. This can be generalized to tensor fields; in fact, you just append indices to $\phi$.

*Remark* 3. In general, $\mathscr{L}$ will depend on higher order gradients. Our action depends on gradients of $A^\mu(x)$ by Axiom 3(b).

*Remark* 4. Our E-L equations for fields are PDEs, in contrast to mechanics where we only had coupled ODEs!

[2] We use the notation $d\boldsymbol{x}$ to represent the volume element for $\mathbb{R}^3$ (in Cartesian coordinates, $d\boldsymbol{x} = dx\,dy\,dz$.

### 1.3 The field equations

From Axiom 3(b), the Maxwellian Lagrangian density is

$$\boxed{\mathscr{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu}}$$
$$= \mathscr{L}\left(A^{\mu}(x), \partial^{\nu} A^{\mu}(x)\right).$$

Therefore, our Euler-Lagrange system of equations (Equation (2.1)) becomes

$$\partial_{\beta} \frac{\partial \mathscr{L}}{\partial\left(\partial_{\beta} A_{\alpha}(x)\right)} = \frac{\partial \mathscr{L}}{\partial\left(A_{\alpha}(x)\right)}. \tag{2.2}$$

Now, $F^{\mu\nu}$ is defined in terms of gradients of $A^{\mu}$ only, so

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial\left(A_{\alpha}(x)\right)} &= -\frac{1}{c} \frac{\partial}{\partial\left(A_{\alpha}(x)\right)}\left[J^{\mu}(x) A_{\mu}(x)\right] \\ &= -\frac{1}{c} J^{\alpha}(x). \end{aligned} \tag{2.3}$$

On the other side of the equation,

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial\left(\partial_{\beta} A_{\alpha}(x)\right)} &= -\frac{1}{16\pi} \frac{\partial}{\partial\left(\partial_{\beta} A_{\alpha}\right)}\left[F_{\mu\nu} g^{\kappa\mu} g^{\lambda\nu} F_{\kappa\lambda}\right] \\ &= -\frac{1}{16\pi} g^{\kappa\mu} g^{\lambda\nu} \frac{\partial}{\partial\left(\partial_{\beta} A_{\alpha}\right)}\left[\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)\left(\partial_{\kappa} A_{\lambda}-\partial_{\lambda} A_{\kappa}\right)\right] \\ &= -\frac{1}{16\pi} g^{\kappa\mu} g^{\lambda\nu}\left[\left(\delta_{\mu}^{\beta} \delta_{\nu}^{\alpha}-\delta_{\nu}^{\beta} \delta_{\mu}^{\alpha}\right)\left(\partial_{\kappa} A_{\lambda}-\partial_{\lambda} A_{\kappa}\right)+\left(\delta_{\kappa}^{\beta} \delta_{\lambda}^{\alpha}-\delta_{\lambda}^{\beta} \delta_{\kappa}^{\alpha}\right)\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)\right] \\ &= -\frac{1}{16\pi}\left[\left(g^{\kappa\beta} g^{\lambda\alpha}-g^{\kappa\alpha} g^{\lambda\beta}\right)\left(\partial_{\kappa} A_{\lambda}-\partial_{\lambda} A_{\kappa}\right)+\left(g^{\beta\mu} g^{\alpha\nu}-g^{\alpha\mu} g^{\beta\nu}\right)\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)\right] \\ &= -\frac{1}{16\pi}\left[\left(\partial^{\beta} A^{\alpha}-\partial^{\alpha} A^{\beta}-\partial^{\alpha} A^{\beta}+\partial^{\beta} A^{\alpha}\right)+\left(\partial^{\beta} A^{\alpha}-\partial^{\alpha} A^{\beta}-\partial^{\alpha} A^{\beta}+\partial^{\beta} A^{\alpha}\right)\right] \\ &= -\frac{1}{16\pi}\left[4\left(\partial^{\beta} A^{\alpha}-\partial^{\alpha} A^{\beta}\right)\right] \\ &= -\frac{1}{4\pi} F^{\beta\alpha}. \end{aligned} \tag{2.4}$$

Inserting Equations (2.3) and (2.4) into our Euler Lagrange Equation (2.2), we obtain

$$\boxed{\boxed{\partial_{\mu} F^{\mu\nu}(x) = \frac{4\pi}{c} J^{\nu}(x)}}. \tag{2.5}$$

*Remark* 1. All physical fields must obey these four equations.

*Remark* 2. Since $F^{\mu\nu}$ is defined in terms of $A^{\mu}$, these equations are differential equations for $A^{\mu}$, making $A^{\mu}$ the "fundamental" physical object. Alternatively, we can augment Equations (2.5) by Proposition 1:

$$\boxed{\boxed{\partial_{\mu} \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda}(x) = 0}}, \tag{2.6}$$

which contains the structure of $F^{\mu\nu}$ in terms of gradients of $A^{\mu}$. We can then consider Equations (2.5) and (2.6) to be field equations for $F^{\mu\nu}$, regarding $F^{\mu\nu}$ as fundamental.

## 2 Conservation laws and gauge invariance

### 2.1 Continuity equation for the 4-current

**Proposition 1.** *The 4-current obeys the* continuity equation*:*

$$\boxed{\partial_\mu J^\mu(x) = 0}.$$

*Proof.* From § 1.3, Equation (2.5),

$$\partial_\nu J^\nu = \frac{c}{4\pi}\underbrace{\partial_\nu\partial_\mu}_{\text{sym.}}\underbrace{F^{\mu\nu}}_{\text{a-sym.}} = -\frac{c}{4\pi}\underbrace{\partial_\mu\partial_\nu F^{\nu\mu}}_{\text{relabel }\mu\leftrightarrow\nu} = -\frac{c}{4\pi}\partial_\nu\partial_\mu F^{\mu\nu} = -\partial_\nu J^\nu$$

$$\implies \quad \partial_\nu J^\nu = 0$$

□

*Remark* 1. The 4-vector $J^\mu = \left(J^0, \boldsymbol{J}\right)$ has a time-like component defined as $J^0 =: c\rho$ and space-like component defined as $\boldsymbol{J} =: \boldsymbol{j}$. That is,

$$\boxed{J^\mu =: (c\rho, \boldsymbol{j})}.$$

$\rho$ is called *electric charge density* and $\boldsymbol{j}$ is called *electric current density.*

*Remark* 2. In terms of $\rho$ and $\boldsymbol{j}$, Proposition 1 takes the form $c\partial_0\rho + \partial_i j^i = 0$. But $\partial_0 = \frac{\partial}{\partial(ct)} = \frac{1}{c}\partial_t$ and $\partial_i = \frac{\partial}{\partial x^i} =: \nabla_i$; thus, the continuity equation is equivalent to

$$\boxed{\partial_t\rho(\boldsymbol{x}, t) + \nabla\cdot\boldsymbol{j}(\boldsymbol{x}, t) = 0}. \tag{2.7}$$

*Remark* 3. Integrate Equation (2.7) over a spatial volume $V$ with surface boundary $(V)$:

$$\partial_t\int_V d^3x\,\rho(\boldsymbol{x}, t) = -\int_V d^3x\,\nabla\cdot\boldsymbol{j}(\boldsymbol{x}, t) = -\int_{(V)} d\boldsymbol{S}\cdot\boldsymbol{j}(\boldsymbol{x}, t).$$

Define

$$\boxed{Q(t) := \int_V d^3x\,\rho(\boldsymbol{x}, t)}$$

to be the *total charge* within $V$. Then

$$\frac{dQ}{dt} = -\int_{(V)} d\boldsymbol{S}\cdot\boldsymbol{j}.$$

In words, the total charge within $V$ can only change if there is a flux of charge current through the boundary surface $(V)$, hence the name "continuity equation".[3]

## 2.2 The energy-momentum tensor

**Definition 1.** ***Electromagnetic energy-momentum tensor.*** The tensor field $T^{\mu\nu}(x)$, defined as

$$\boxed{T^{\mu\nu} := -\frac{1}{4\pi}F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}},$$

is called the *electromagnetic energy-momentum tensor.*

*Remark* 1. It is not obvious what this tensor has to do with energy and momentum for now; see Problem #16 for some hints and LL for details.

[3] This "conservation of charge" is a result of our field equations. The field equations are in turn a result of the actions we have postulated through axioms.

**Proposition 1.**

**(1)** $T^{\mu\nu}$ *is symmetric;* $\boxed{T^{\mu\nu}(x) = T^{\nu\mu}(x)}$.

**(2)** $T^{\mu\nu}$ *is traceless;* $\boxed{T_\mu{}^\mu(x) = 0}$.

*Proof.*

**(1)** We know the second term in the definition of $T^{\mu\nu}$ is symmetric. For the first term,

$$F^{\mu\alpha}F^\nu{}_\alpha = g^{\alpha\beta}F^\mu{}_\beta g_{\alpha\gamma}F^{\nu\gamma} = \delta^\beta_\gamma F^\mu{}_\beta F^{\nu\gamma} = F^{\nu\beta}F^\mu{}_\beta.$$

Thus the first term is symmetric and, in turn, $T^{\mu\nu}$ is symmetric.

**(2)** $-4\pi T^\mu{}_\mu = F^{\mu\alpha}F_{\mu\alpha} - \frac{1}{4}\underbrace{g^\mu{}_\mu}_{\delta^\mu_\mu=4}F_{\alpha\beta}F^{\alpha\beta} = 0$ $\square$

*Remark* 2. By Ch. 1, § 3.4, $T^{\mu\nu}$ can be decomposed into $T^{00}$, Euclidean vector $T^{0j}$, plus symmetric Euclidean tensor $T^{jk}$.

## 2.3 The continuity equation for the energy-momentum tensor

**Proposition 1.** *In the absence of matter ($J^\mu = 0$), $T^{\mu\nu}$ obeys*

$$\partial_\nu T_\mu{}^\nu(x) = 0.$$

*Proof.* From Definition 1,[a]

$$\begin{aligned}\partial_\nu T_\mu{}^\nu &= \frac{1}{4\pi}\left[-\partial_\nu F_\mu{}^\alpha F^\nu{}_\alpha + \frac{1}{4}\partial_\nu\delta^\nu_\mu F_{\alpha\beta}F^{\alpha\beta}\right]\\ &= \frac{1}{4\pi}\left[-\left(\partial_\nu F_\mu{}^\alpha\right)F^\nu{}_\alpha - F_\mu{}^\alpha\partial_\nu F^\nu{}_\alpha + \frac{1}{4}\partial_\mu F_{\alpha\beta}F^{\alpha\beta}\right].\end{aligned}$$

But by Equation (2.5), $\partial_\nu F^\nu{}_\alpha = \frac{4\pi}{c}J_\alpha = 0$. Furthermore, the last term can be rewritten as follows:

$$\begin{aligned}F_{\alpha\beta}\partial_\mu F^{\alpha\beta} &= g_{\alpha\gamma}g_{\beta\kappa}F^{\gamma\kappa}\partial_\mu g^{\alpha\epsilon}g^{\beta\nu}F_{\epsilon\nu} = \delta^\nu_\kappa\delta^\epsilon_\gamma F^{\gamma\kappa}\partial_\mu F_{\epsilon\nu} = F^{\epsilon\kappa}\partial_\mu F_{\epsilon\kappa}\\ &= \left(\partial_\mu F_{\alpha\beta}\right)F^{\alpha\beta}\end{aligned}$$

$$\implies \quad \partial_\mu F_{\alpha\beta}F^{\alpha\beta} = 2\left(\partial_\mu F_{\alpha\beta}\right)F^{\alpha\beta}.$$

$$\implies \quad \partial_\nu T_\mu{}^\nu = \frac{1}{4\pi}\left[-\left(\partial_\nu F_\mu{}^\alpha\right)F^\nu{}_\alpha + \frac{1}{2}\left(\partial_\mu F_{\alpha\beta}\right)F^{\alpha\beta}\right].$$

By Problem #12 (see Belitz's solution),

$$0 = \partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha}.$$

$$\begin{aligned}
\implies \partial_\nu T_\mu{}^\nu &= \frac{1}{4\pi}\left[-\left(\partial_\nu F_\mu{}^\alpha\right)F^\nu{}_\alpha - \frac{1}{2}\left(\partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha}\right)F^{\alpha\beta}\right] \\
&= \frac{1}{4\pi}\left[-\left(\partial_\nu F_{\mu\alpha}\right)F^{\nu\alpha} + \frac{1}{2}\underbrace{\left(\partial_\alpha F_{\mu\beta}\right)F^{\alpha\beta}}_{\alpha\to\nu,\,\beta\to\alpha} + \frac{1}{2}\underbrace{\left(\partial_\beta F_{\mu\alpha}\right)F^{\beta\alpha}}_{\beta\to\nu}\right]. \\
&= \frac{1}{4\pi}\left[-\left(\partial_\nu F_{\mu\alpha}\right)F^{\nu\alpha} + \frac{1}{2}\left(\partial_\nu F_{\mu\alpha}\right)F^{\nu\alpha} + \frac{1}{2}\left(\partial_\nu F_{\mu\alpha}\right)F^{\nu\alpha}\right] \\
&= 0
\end{aligned}$$

Note that in the third to last line we used the identity that, for any tensor contraction,

$$t^{(\cdots)\alpha(\cdots)}{}_\alpha{}^{(\cdots)} = t^{(\cdots)}{}_\alpha{}^{(\cdots)\alpha(\cdots)}.$$

That is, contracted indices can swap being upstairs or downstairs. □

[a] In the first line, we use the notational convention that $\partial_\nu F_\mu{}^\alpha F^\nu{}_\alpha$ implies that the partial acts on everything to the right; that is, $\partial_\nu F_\mu{}^\alpha F^\nu{}_\alpha := \partial_\nu\left(F_\mu{}^\alpha F^\nu{}_\alpha\right)$. If we wanted the partial only acting on $F_\mu{}^\alpha$, we would have to write $\left(\partial_\nu F_\mu{}^\alpha\right)F^\nu{}_\alpha$.

*Remark* 1. For any rank-$(n+1)$ tensor field $t^{\mu\alpha_1\cdots\alpha_n}(x)$, the continuity equation $\partial_\mu t^{\mu\alpha_1\cdots\alpha_n} = 0$ implies a conservation law for the rank-$n$ tensor $t^{0\alpha_1\cdots\alpha_n}(x)$ by the arguments from § 2.1. $\partial_\mu J^\mu = 0$ is the case where $n = 0$; the proposition above is the case where $n = 1$.

**Corollary 1.** *In the presence of matter, the continuity equation gets modified to*

$$\boxed{\partial_\nu T_\mu{}^\nu = -\frac{1}{c}F_\mu{}^\nu J_\nu}.$$

*Proof.* Problem #17. □

## 2.4 Gauge invariance

Let $\chi(x)$ be an arbitrary scalar function of spacetime.

**Definition 1.** ***Gauge transformation.*** A transformation of the potential $A^\mu(x)$ according to

$$\boxed{A^\mu \to A^\mu - \partial^\mu\chi}$$

is called a *gauge transformation.*

**Proposition 1.** *The action from Axiom 3 is invariant under gauge transformations.*

*Proof.* $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \to \partial^\mu A^\nu - \partial^\mu\partial^\nu\chi - \partial^\nu A^\mu + \partial^\nu\partial^\mu\chi = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$, so $S_{\text{vac}}$ is invariant.

$$S_{\text{int}} = -\tfrac{1}{c}\int d^4x J_\mu A^\mu \to S_{\text{int}} \underbrace{- \frac{1}{c}\int d^4x J_\mu\partial^\mu\chi}_{\text{integ. by parts}} = S_{\text{int}} + \tfrac{1}{c}\int d^4x \underbrace{\left(\partial^\mu J_\mu\right)}_{=0}\chi = S_{\text{int}},$$ so $S_{\text{int}}$ is invariant.

Therefore the total action is invariant. □

*Remark* 1. The potential is not unique. This is a result of the fact that $F^{\mu\nu}$ depends only on gradients of $A^\mu$.

*Remark* 2. We may choose a gauge transformation to enforce a particular condition on $A^\mu$.

**Corollary 1.** *$A^{\mu}(x)$ can always be chosen (gauge transformed) such that*

$$\partial_{\mu}A^{\mu}=0,$$

*called the* Lorenz gauge.

*Proof.* Choose $\chi(x)$ such that it solves the PDE $\partial^{\mu}\partial_{\mu}\chi=\partial_{\mu}A^{\mu}$ (Laplace's equation).

$$\Longrightarrow \qquad \partial_{\mu}A^{\mu}\to\partial_{\mu}A^{\mu}-\underbrace{\partial_{\mu}\partial^{\mu}\chi}_{=\partial_{\mu}A^{\mu}}=0.$$

□

*Remark* 3. $\partial_{\mu}A^{\mu}$ is a Lore**ntz** scalar, and so the Lore**nz** gauge is Lore**ntz** invariant.

# 3 Electric and magnetic fields

## 3.1 The field tensor in terms of Euclidean vector fields

Since $F^{\mu\nu}$ is an antisymmetric Minkowski tensor, from Ch. 1, § 3.4 we can write it in the form

$$\begin{aligned}F_{\mu\nu} &= \left(\begin{array}{c:ccc}0 & E_x & E_y & E_z\\ \hdashline -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_x & -B_y & B_x & 0\end{array}\right)\\ &= \left(\begin{array}{c:c}0 & \boldsymbol{E}\\ \hdashline -\boldsymbol{E} & B_{jk}\end{array}\right)\end{aligned}$$

...with $\boldsymbol{E}(x)=(E_x(x),E_y(x),E_z(x))$ a Euclidean vector field,

...and $\boldsymbol{B}(x)=(B_x(x),B_y(x),B_z(x))$ a Euclidean pseudovector field.

Beware! There are some subtle notational details here. The above definition uses Landau & Lifshitz's notation. In terms of the numerical indices used throughout the rest of this text, these vector fields are

$$\begin{aligned}\boldsymbol{E} &= (E_x,E_y,E_z):=\left(E^1,E^2,E^3\right)=-(E_1,E_2,E_3)\\ \boldsymbol{B} &= (B_x,B_y,B_z):=\left(B^1,B^2,B^3\right)=-(B_1,B_2,B_3).\end{aligned}$$

One must be careful to not identify $E_x$ with $E_1$!

**Definition 1.** ***Electric and magnetic field; magnetic field tensor.***
**(a)** $\boldsymbol{E}(x)$ is called *electric field*; $\boldsymbol{B}(x)$ is called *magnetic field.*
**(b)** The antisymmetric Euclidean tensor

$$B_{jk}=\begin{pmatrix}0 & -B_z & B_y\\ B_z & 0 & -B_x\\ -B_y & B_x & 0\end{pmatrix}=B^{jk}=-\varepsilon^{jkl}B_l$$

is called *magnetic field tensor.*

*Remark* 1. What about the contravariant components $F^{\mu\nu}$?

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{\mu} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{\nu} F_{\mu\nu}$$

$$= \left( \begin{array}{c|c} 0 & -\boldsymbol{E} \\ \hline \boldsymbol{E} & B^{jk} = B_{jk} \end{array} \right)$$

From Ch. 1 § 3.4, we can write $A^\mu$ in the form

$$A^\mu(x) = (\phi(x), \boldsymbol{A}(x)),$$

...with $\phi(x) := A^0(x) = A_0(x)$ a Euclidean scalar field,
...and $\boldsymbol{A}(x) = \left(A^1(x), A^2(x), A^3(x)\right)$ a Euclidean vector field.

**Definition 2.** ***Scalar and vector potential.*** $\phi(x)$ is called *scalar potential*, and $\boldsymbol{A}(x)$ is called *vector potential.*

*Remark* 2. This is analogous to $J^\mu(x) = (c\rho(x), \boldsymbol{j}(x))$, with $\rho$ the charge density and $\boldsymbol{j}$ the current density (see § 2.1).

## 3.2 Maxwell's equations

From § 1.3 Equation (2.6), we have

$$\partial_\mu \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = 0.$$

What are these in terms of $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$?

**Proposition 1.** *The field equation*

$$\partial_\mu \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = 0$$

*is equivalent to*[a]

$$\boxed{\boxed{\nabla \cdot \boldsymbol{B} = 0}} \tag{M1}$$

$$\boxed{\boxed{\frac{1}{c}\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = \boldsymbol{0}}} \tag{M2}$$

[a] Belitz refers to these as (1) and (2) instead of $(M1)$ and $(M2)$, respectively.

*Proof.*
$\underline{\nu = 0}$: Note that for, say, $\mu = 1$, we obtain $\partial_1 \varepsilon^{10\kappa\lambda} F_{\kappa\lambda} = -\partial_1 F_{23} + \partial_1 F_{32} = -2\partial_1 F_{23}$. Thus,

$$0 = 2\left(-\partial_1 F_{23} - \partial_2 F_{31} - \partial_3 F_{12}\right) = 2\underbrace{(\partial_1 B_x + \partial_2 B_y + \partial_3 B_z)}_{\nabla\cdot\boldsymbol{B}} \implies \nabla \cdot \boldsymbol{B} = 0$$

$\underline{\nu = 1}$: Again, for a given choice of $\mu$, the above simplification applies. Thus,

$$0 = 2\left(\partial_0 F_{23} - \partial_2 F_{03} + \partial_3 F_{02}\right) = 2(-\frac{1}{c}\partial_t B_x \underbrace{-\partial_2 E_z + \partial_3 E_y}_{-(\nabla\times\boldsymbol{E})_x}) \implies \frac{1}{c}\partial_t B_x + (\nabla \times \boldsymbol{E})_x = 0$$

$\underline{\nu = 2, 3}$: Cyclically permute the $\nu = 1$ case. $\square$

*Remark* 1. These are the four homogeneous PDEs known as the first two Maxwell Equations.

Now consider, from § 1.3, the Euler-Lagrange equation (2.5):

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu.$$

What are these in terms of $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$?

**Proposition 2.** *The field equation*

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

*is equivalent to*

$$\boxed{\boxed{\nabla \cdot \boldsymbol{E} = 4\pi\rho}} \tag{M3}$$

$$\boxed{\boxed{-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}}} \tag{M4}$$

*Proof.*

$\underline{\nu = 0}$: $\underbrace{\partial_0 F^{00}}_{0} + \underbrace{\partial_j F^{j0}}_{\nabla\cdot\boldsymbol{E}} = \frac{4\pi}{c}\underbrace{J^0}_{c\rho} \implies \nabla \cdot \boldsymbol{E} = 4\pi\rho$

$\underline{\nu = 1}$: $\partial_0 F^{01} + \partial_i F^{i1} = \frac{4\pi}{c} J^1 \implies -\frac{1}{c}\partial_t E_x + \partial_2 B_z - \partial_3 B_y = -\frac{1}{c}\partial_t E_x + (\nabla \times \boldsymbol{B})_x = \frac{4\pi}{c} j_x.$

$\underline{\nu = 2, 3}$: Cyclically permute the $\nu = 1$ case. □

*Remark* 2. Equations (M1)-(M4) are called *Maxwell's Equations.* Their solutions determine physical field equations for given charge and current densities.

*Remark* 3. Equations (M1)-(M4) are equivalent to Equations (2.6) and (2.5).

*Remark* 4. $\boldsymbol{E}$ and $\boldsymbol{B}$ are Euclidean vector fields, so the Lorentz invariance thereof is obscured.

*Remark* 5. *Units*: We use CGS (centimeter-gram-second) units, not SI units (see table below). At some point when I have time I will write Maxwell's equations in SI units here.

| [unit] | CGS | SI |
|---|---|---|
| [charge] | esu $= \mathrm{g}^{1/2}\,\mathrm{cm}^{3/2}\,\mathrm{s}^{-1}$ | C |
| $[\rho]$ | $\mathrm{g}^{1/2}\,\mathrm{cm}^{-3/2}\,\mathrm{s}^{-1}$ | $\mathrm{C\,m}^{-3}$ |
| $[\boldsymbol{j}]$ | $\mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-2}$ | $\mathrm{C\,m}^{-2}\,\mathrm{s}^{-1}$ |
| $[\boldsymbol{E}]$ | $\mathrm{g}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{s}^{-1}$ | $\mathrm{N\,C}^{-1}$ |
| $[\boldsymbol{B}]$ | gauss $= \mathrm{g}^{1/2}\,\mathrm{cm}^{1/2}\,\mathrm{s}^{-1}$ $= \mathrm{esu\,cm}^{-2}$ | $\mathrm{N\,A}^{-1}\,\mathrm{m}^{-1}$ |

Table 2.1: Comparison of CGS and SI units.

## 3.3 Discussion of Maxwell's equations

### Gauss' Law

Consider a localized charge density $\rho$ in a larger volume $V$ with boundary surface $(V)$. Integrate (M3) over $V$:

$$\int_V d^3x\, \nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = 4\pi \int_V d^3x\, \rho(\boldsymbol{x}, t).$$

If we define the *total charge* within $V$ to be

$$\boxed{Q(t) := \int_V d^3x\, \rho(\boldsymbol{x}, t)}$$

then by using this and the Divergence Theorem above, we obtain

$$\boxed{\Phi_{\boldsymbol{E}} := \int_{(V)} d\boldsymbol{S} \cdot \boldsymbol{E} = 4\pi Q}$$

In words, the flux of electric field through a closed surface is equal to the total charge contained therein.

*Remark* 1. This is called *Gauss' Law*.

*Remark* 2. Electric charges are the sources of electric fields.

### Magnetic field divergence

Integrate (M1) over $V$:

$$0 = \int_V d^3x\, \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t).$$

Again, the Divergence Theorem yields

$$\boxed{\Phi_{\boldsymbol{B}} := \int_{(V)} d\boldsymbol{S} \cdot \boldsymbol{B} = 0}.$$

In words, the flux of magnetic field through a closed surface is always zero.

*Remark* 3. The magnetic field has no sources; there are no magnetic monopoles.

*Remark* 4. In our Lorentz invariant formulation of 1, this comes from the asymmetry of the two field equations:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0.$$

### Faraday's Law

Consider a surface $S$ with boundary $(S)$.

Integrate (M2) over $S$:

$$-\frac{1}{c} \int_S d\boldsymbol{S} \cdot \partial_t \boldsymbol{B}(\boldsymbol{x}, t) = \int_S d\boldsymbol{S} \cdot (\nabla \times \boldsymbol{E})(\boldsymbol{x}, t).$$

By Stokes' Theorem,

$$\boxed{-\dot{\Phi}_{\boldsymbol{B}} = c \oint_{(S)} d\boldsymbol{l} \cdot \boldsymbol{E}}$$

In words, the circulation of the electric field around a loop is proportional to the time rate of change of the magnetic flux through a surface bounded by that loop.

*Remark* 5. This is called *Faraday's Law of induction.*

*Remark* 6. Consider a closed $\boldsymbol{E}-$field line. $\oint d\boldsymbol{l} \cdot \boldsymbol{E} > 0 \implies \dot{\Phi}_{\boldsymbol{B}} < 0$. So in a static $\boldsymbol{B}-$field, there can be *no* closed $\boldsymbol{E}-$field lines!

**Ampère-Maxwell Law**

Integrate (M4) over $S$:

$$\int_S d\boldsymbol{S} \cdot (\nabla \times \boldsymbol{B})(\boldsymbol{x}, t) = \frac{4\pi}{c} \int_S d\boldsymbol{S} \cdot \boldsymbol{J}(\boldsymbol{x}, t) + \frac{1}{c} \int_S d\boldsymbol{S} \cdot \partial_t \boldsymbol{E}(\boldsymbol{x}, t).$$

We define the *total current* to be

$$\boxed{I(t) := \int_S d\boldsymbol{S} \cdot \boldsymbol{J}(\boldsymbol{x}, t)}.$$

Using Stokes' Law once more yields

$$\boxed{c \oint_{(S)} d\boldsymbol{l} \cdot \boldsymbol{B} = 4\pi I + \dot{\Phi}_{\boldsymbol{E}}}$$

In words, the circulation of the magnetic field around a loop is proportional to the sum of the total current and the *displacement current*.

*Remark* 7. This is called the *Ampère-Maxwell Law.*

*Remark* 8. Currents induce $\boldsymbol{B}-$fields, and vice versa.

*Remark* 9. For static fields, we have Ampère's Law:

$$c \oint d\boldsymbol{l} \cdot \boldsymbol{B} = 4\pi I.$$

The displacement current was later added by Maxwell.

## 3.4 Relations between fields and potentials

*Claim* 1. The electric and magnetic fields are related to the 4-potential by

$$\boxed{\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\partial_t \boldsymbol{A}}$$

$$\boxed{\boldsymbol{B} = \nabla \times \boldsymbol{A}}$$

*Proof.* From § 3.1,

$$E^j = -F^{0j} = -\partial^0 A^j + \partial^j A^0 = -\partial_0 A^j - \partial_j A^0 = -\frac{1}{c}\partial_t A^j - \partial_j \phi.$$

We also determined

$$F_{12} = -B^3 = \partial_1 A_2 - \partial_2 A_1 = (\nabla \times \boldsymbol{A})_3 = -(\nabla \times \boldsymbol{A})^3,$$

and cyclic for $B^2$, $B^1$. □

*Remark* 1. In general, both the scalar and vector potentials determine $\boldsymbol{E}$.

*Remark* 2. As a safety check, lets try gauge transforming the relation for $\boldsymbol{E}$:

$$A^\mu \to A^\mu - \partial^\mu \chi \quad \Longrightarrow \quad \begin{cases} \phi \to & \phi - \frac{1}{c}\partial_t \chi \\ \boldsymbol{A} \to & \boldsymbol{A} + \nabla \chi \end{cases}$$

$$\Longrightarrow \boldsymbol{E} \to \boldsymbol{E} + \nabla \frac{1}{c}\partial_t \chi - \frac{1}{c}\partial_t \nabla \chi = \boldsymbol{E}.$$

Thus, $\boldsymbol{E}$ is invariant.

*Remark* 3. $\boldsymbol{B}$ is also invariant under gauge transformations since $\nabla \times (\nabla \chi) = 0$.

## 3.5 Charges in electromagnetic fields

So far, our attitude has been that the field equations determine $\boldsymbol{E}$ and $\boldsymbol{B}$ for given charges and currents. What about the converse? For given fields, what is their influence on a point charge?

Let a point particle with charge $e$ by at point $\boldsymbol{y}(t)$ with velocity $\dot{\boldsymbol{y}}(t) =: \boldsymbol{v}(t)$.

**charge density** $\cdots$ $\rho(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{y}(t))$

**current density** $\cdots$ $\boldsymbol{j}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\, \boldsymbol{v}(t)$

**4-current** $\cdots$ $J^\mu = (c\rho, \boldsymbol{j})\,,\ J_\mu = (c\rho, -\boldsymbol{j})$

**4-potential** $\cdots$ $A^\mu = (\phi, \boldsymbol{A})$

By Axiom 3,

$$\begin{aligned} S_{\text{int}} &= -\frac{1}{c}\int d^4x\, J_\mu(x)\, A^\mu(x) \\ &= -\frac{1}{c}\int dt \int d\boldsymbol{x}\, c\rho(\boldsymbol{x}, t)\, \phi(\boldsymbol{x}, t) + \frac{1}{c}\int dt \int d\boldsymbol{x}\, \boldsymbol{j}(\boldsymbol{x}, t) \cdot \boldsymbol{A}(\boldsymbol{x}, t) \\ &= -e\int dt\, \phi(\boldsymbol{y}, t) + \frac{e}{c}\int dt\, \boldsymbol{v}(t) \cdot \boldsymbol{A}(\boldsymbol{y}, t)\,. \end{aligned}$$

Now consider the Lagrangian of the point particle, $\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{int}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$, which is related to $S_{\text{int}}$ via $S_{\text{int}} = \int dt \mathcal{L}_{\text{int}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)$. Comparison with the above equation reveals

$$\boxed{\mathcal{L}_{\text{int}}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = -e\phi(\boldsymbol{y}, t) + \frac{e}{c}\boldsymbol{v}(t) \cdot \boldsymbol{A}(\boldsymbol{y}, t)}\,.$$

*Remark* 1. These are the scalar and vector potentials from PHYS611 Ch2 1.3 Example 1!

*Remark* 2. Axiom 3 is consistent with our Mechanics axioms.

*Remark* 3. $\mathcal{L}_{\text{int}}$ must be augmented by the free particle Lagrangian $\mathcal{L}_0$. Since the field equations are Lorentz invariant, we must pick $\mathcal{L}_0$ such that it is as well; we need the Einsteinian $\mathcal{L}_0^E$ for consistency. However, the Galilean $\mathcal{L}_0^G$ works well enough if $|\boldsymbol{v}| \ll c$.

*Remark* 4. The momentum of the particle is $\boldsymbol{p}=\frac{\partial\mathcal{L}_0}{\partial\boldsymbol{v}}$ (not $\frac{\partial\mathcal{L}}{\partial\boldsymbol{v}}$; see PHYS611 Ch2 1.4), and Newton's 2nd Law takes the form (from PHYS611):

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{p}=\boldsymbol{F} &= \boldsymbol{F}^{(1)}+\boldsymbol{F}^{(2)} \\
&= -\nabla\underbrace{e\phi}_{U}-(\partial_t-\boldsymbol{v}\times)\underbrace{\left(\frac{e}{c}\boldsymbol{A}\right)}_{\boldsymbol{V}} \\
&= e\underbrace{(-\nabla\phi-\frac{1}{c}\partial_t\boldsymbol{A})}_{\boldsymbol{E}}+\frac{e}{c}\boldsymbol{v}\times\underbrace{(\nabla\times\boldsymbol{A})}_{\boldsymbol{B}}
\end{aligned}$$

$$\Longrightarrow \boxed{\frac{d}{dt}\boldsymbol{p}=\boldsymbol{F}=e\boldsymbol{E}+\frac{e}{c}\boldsymbol{v}\times\boldsymbol{B}}$$

which is the electric force plus the Lorentz force! In conclusion, all of this is consistent with what we did in Mechanics.

## 3.6 Poynting's theorem

Consider the continuity equation for the energy-momentum tensor

$$\partial_\nu T^{\mu\nu}=-\frac{1}{c}F^{\mu\nu}J_\nu$$

from § 2.3 for $\mu=0$:[4]

$$\underline{T^{00}}=-\frac{1}{8\pi}\left[2F^{0\alpha}F^{0}{}_{\alpha}-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\right]=-\frac{1}{8\pi}\left[2\boldsymbol{E}^2-\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right)\right]=\underline{-\frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)};$$

$$\underline{T^{01}}=-\frac{1}{4\pi}F^{0\alpha}F^{1}{}_{\alpha}+0=-\frac{1}{4\pi}F^{0j}F^{1}{}_{j}=-\frac{1}{4\pi}\left(E_yB_z-E_zB_y\right)=\underline{-\frac{1}{4\pi}\left(\boldsymbol{E}\times\boldsymbol{B}\right)^1},$$

and cyclic.

$$\begin{aligned}
\Longrightarrow \partial_\nu T^{0\nu} &= \frac{1}{c}\partial_t\left[-\frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right]+\nabla\cdot\left(-\frac{1}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\right) \\
&= - F^{0\nu}J_\nu=-F^{0j}J_j=\boldsymbol{E}\cdot\boldsymbol{j}.
\end{aligned}$$

We summarize this with some new definitions as follows.

*Claim* 1. ***Poynting's theorem.*** Define the *energy density of the fields* $u\left(\boldsymbol{x},t\right)$ as

$$\boxed{u:=\frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)},$$

and define the *energy current density* (or *Poynting vector*) $\boldsymbol{P}\left(\boldsymbol{x},t\right)$ as

$$\boxed{\boldsymbol{P}:=\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}}.$$

Then

$$\boxed{\partial_t u+\nabla\cdot\boldsymbol{P}=-\boldsymbol{E}\cdot\boldsymbol{j}}$$

[4] Recall that $T^{\mu\nu}:=-\frac{1}{4\pi}F^{\mu\alpha}F^{\nu}{}_{\alpha}+\frac{1}{16\pi}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ from § 2.2, and $F_{\alpha\beta}F^{\alpha\beta}=2\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right)$ from Ch. 1 § 3.4.

*Proof.* Follows directly from the above discussion. $\square$

*Remark* 1. For $\boldsymbol{j} = \boldsymbol{0}$, this expresses local energy conservation. It is analogous to § 2.1 with $\rho \to u$, $\boldsymbol{j} \to \boldsymbol{P}$.

*Remark* 2. Recall from § 3.5 that since $\boldsymbol{F} = e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B}$, the work per unit time (power) done by the fields on a charge $e$ is $\boldsymbol{v} \cdot \boldsymbol{F} = e\boldsymbol{v} \cdot \boldsymbol{E}$. This implies

$$\boldsymbol{j} \cdot \boldsymbol{E} = \left(\frac{e\boldsymbol{v}}{V}\right) \cdot \boldsymbol{E} = \frac{\boldsymbol{v} \cdot \boldsymbol{F}}{V}$$

is the work per unit time and volume, or power density. So for $\boldsymbol{j} = \boldsymbol{0}$, Poynting's theorem still expresses energy conservation. In words,

$$\begin{aligned} (\text{energy change}) \quad &= \quad -\,(\text{energy transported by the energy current}) \\ &\quad\;\; -\,(\text{work done by the field on the charges}) \end{aligned}$$

*Remark* 3. We still need to show that $u(\boldsymbol{x}, t)$ can be sensibly interpreted as the energy density of the field.

Let $\boldsymbol{j}(\boldsymbol{x}, t)$ be the current density due to just one particle, as in § 3.5 (for many particles, we just sum over them). Integrating,

$$\int d\boldsymbol{x}\, \boldsymbol{j} \cdot \boldsymbol{E} = \int d\boldsymbol{x}\ [\boldsymbol{E}(\boldsymbol{x}, t)] \cdot [e\boldsymbol{v}\delta(\boldsymbol{x} - \boldsymbol{y})] = e\boldsymbol{v} \cdot \boldsymbol{E}(\boldsymbol{y}),$$

where $\boldsymbol{y}$ is the position of the particle.

Consider a non-relativistic particle:

$$\begin{aligned} E_{kin} &= \frac{m}{2}\boldsymbol{v}^2 \\ \Longrightarrow \frac{d}{dt}E_{kin} &= m\boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} = \boldsymbol{v} \cdot \underbrace{\frac{d}{dt}\boldsymbol{p}}_{=e\boldsymbol{E}} = e\boldsymbol{v} \cdot \boldsymbol{E}, \end{aligned}$$

where the last step follows from § 3.5. Now, integrating Poynting's theorem over all space,

$$\frac{d}{dt}\int d\boldsymbol{x}\, u(\boldsymbol{x}, t) + \underbrace{\int d\boldsymbol{x}\, \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, t)}_{\substack{=\int d\boldsymbol{S} \cdot \boldsymbol{P} = 0 \\ (\text{since } \boldsymbol{P} \to \boldsymbol{0} \text{ at } \infty)}} = -\int d\boldsymbol{x}\, \boldsymbol{j} \cdot \boldsymbol{E} = -e\boldsymbol{v} \cdot \boldsymbol{E} = -\frac{d}{dt}E_{kin}.$$

Defining the integral of $u$ as

$$\boxed{U(t) := \int d\boldsymbol{x}\, u(\boldsymbol{x}, t)},$$

we see that

$$\underline{\frac{d}{dt}(U + E_{kin}) = 0.}$$

$U$ must be the *field energy*, since the energy of the particle plus the energy of the field must be conserved. Hence, $u$ is the *energy density of the field.*

*Remark* 4. If we integrated over a finite volume, the energy may change due to an energy current across the volume boundary, and we see that, in general,

$$\int d\boldsymbol{x}\, \nabla \cdot \boldsymbol{P}(\boldsymbol{x}, t) \neq 0.$$

Thus, $\boldsymbol{P}$ should be interpreted as the *energy current density of the field.*

*Remark* 5. The remaining components of the continuity equation from § 2.3,

$$\partial_\nu T^{j\nu} = -\frac{1}{c}F^{j\nu}J_\nu$$

express the fact that the energy current density is also conserved.

# 4 Lorentz transformations of the fields

## 4.1 Physical interpretation of a Lorentz boost

Consider two inertial frames, $CS$ and $\widetilde{CS}$.

Let $\widetilde{CS}$ move with respect to $CS$ with a constant velocity $\boldsymbol{V} = (V, 0, 0)$.

From Problems #8, 10, the transformation from $CS$ to $\widetilde{CS}$ is accomplished by a Lorentz boost:

$$\begin{aligned} c\tilde{t} &= ct\cosh\phi + x\sinh\phi, \\ \tilde{x} &= ct\sinh\phi + x\cosh\phi. \end{aligned}$$

Consider the origin of $\widetilde{CS}$ as viewed by $CS$. Then $\tilde{x} = 0$, and[5]

$$\begin{aligned} x\cosh\phi &= -ct\sinh\phi, \\ \implies V = \frac{x}{t} &= -c\tanh\phi. \end{aligned}$$

$$\implies \sinh\phi = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \frac{V/c}{\sqrt{1-(V/c)^2}}, \quad \cosh\phi = \sqrt{1+\sinh^2\phi} = \frac{1}{\sqrt{1-(V/c)^2}}.$$

*Remark* 1. First, observe that when $c \to \infty$, we recover the Galileo transformation

$$\tilde{x} = x + Vt, \quad \tilde{t} = t.$$

Let us define the above quantities:

$$\boxed{\beta := \frac{V}{c}}, \quad \boxed{\gamma := \frac{1}{\sqrt{1-\beta^2}}} \implies \boxed{\cosh\phi = \gamma}, \quad \boxed{\sinh\phi = \beta\gamma}.$$

With these results, the Lorentz boost can be written

$$\boxed{D^\mu{}_\nu = \begin{pmatrix} \cosh\phi & \sinh\phi & & \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}$$

## 4.2 Transformations of $\boldsymbol{E}$ and $\boldsymbol{B}$ under a Lorentz boost

Consider the field tensor $F^{\mu\nu}$ in $CS$. The field transformed field tensor[6] $\tilde{F}^{\mu\nu}$ in $\widetilde{CS}$ is

$$\tilde{F}^{\mu\nu} = D^\mu{}_\alpha D^\nu{}_\beta F^{\alpha\beta} \quad \text{and} \quad \tilde{x}^\mu = D^\mu{}_\nu x^\nu.$$

[5] I think there's a sign error here.
[6] NOT the dual field tensor

Now let $D^{\mu}{}_{\nu}$ be a Lorentz boost. Then, from § 3.1 we have

$$\begin{aligned}
\tilde{F}^{\mu\nu} &= \left(DFD^T\right)^{\mu\nu} \\
&= \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} E_x\beta\gamma & -E_x\gamma & (-E_y - B_z\beta)\gamma & (-E_z + B_y\beta)\gamma \\ E_x\gamma & -E_x\beta\gamma & (-E_y\beta - B_z)\gamma & (-E_z\beta + B_y)\gamma \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -E_x & -(E_y + B_z\beta)\gamma & -(E_y\beta + B_z)\gamma \\ E_x & 0 & -(E_z - B_y\beta)\gamma & -(E_z\beta - B_y)\gamma \\ (E_y + B_z\beta)\gamma & (E_y\beta + B_z)\gamma & 0 & -B_x \\ (E_z - B_y\beta)\gamma & (E_z\beta - B_y)\gamma & B_x & 0 \end{pmatrix} \\
&=: \begin{pmatrix} 0 & -\tilde{E}_x & -\tilde{E}_y & -\tilde{E}_z \\ \tilde{E}_x & 0 & -\tilde{B}_z & \tilde{B}_y \\ \tilde{E}_y & \tilde{B}_z & 0 & -\tilde{B}_x \\ \tilde{E}_z & -\tilde{B}_y & \tilde{B}_x & 0 \end{pmatrix}
\end{aligned}$$

Thus,

$$\begin{gathered}
\boxed{\tilde{\boldsymbol{E}} \to \begin{cases} \tilde{E}_x = & E_x \\ \tilde{E}_y = & E_y \cosh\phi + B_z \sinh\phi = (E_y + B_z\beta)\gamma \\ \tilde{E}_z = & E_z \cosh\phi - B_y \sinh\phi = (E_z - B_y\beta)\gamma \end{cases}} \\
\boxed{\tilde{\boldsymbol{B}} \to \begin{cases} \tilde{B}_x = & B_x \\ \tilde{B}_y = & B_y \cosh\phi - E_z \sinh\phi = (B_y - E_z\beta)\gamma \\ \tilde{B}_z = & B_z \cosh\phi + E_y \sinh\phi = (B_z + E_y\beta)\gamma \end{cases}}
\end{gathered} \tag{2.8}$$

*Remark* 1. The field equations were formulated in terms of Minkowski tensors; their Lorentz invariance is guaranteed. Equations (2.8) reflect this same Lorentz invariance of Maxwell's equations, which are equivalent to the field equations.

*Remark* 2. Let $V \ll c$, and keep terms to $O\left(\frac{V}{c}\right)$.

$$\implies \cosh\phi \approx 1, \quad \sinh\phi \approx \frac{V}{c}$$

$$\implies \tilde{\boldsymbol{E}} \approx \boldsymbol{E} - \left(\frac{\boldsymbol{V}}{c}\right) \times \boldsymbol{B} + O\left(\left(\frac{V}{c}\right)^2\right), \quad \tilde{\boldsymbol{B}} \approx \boldsymbol{B} + \left(\frac{\boldsymbol{V}}{c}\right) \times \boldsymbol{E} + O\left(\left(\frac{V}{c}\right)^2\right).$$

*Remark* 3. Let $\boldsymbol{E} = 0$, so there is no $\boldsymbol{E}-$field in $CS$.

$$\implies \tilde{\boldsymbol{E}} \approx -\left(\frac{\boldsymbol{V}}{c}\right) \times \boldsymbol{B};$$

we see that in $\widetilde{CS}$ there *is* an $\boldsymbol{E}-$field so long as $\boldsymbol{B} \neq \boldsymbol{0}$!

## 4.3 Lorentz invariants

From the field tensor $F^{\mu\nu}$ we can form the following Lorentz scalar fields:[7]

$$\boxed{I^{(1)} := -\frac{1}{2} F^{\mu\nu} F_{\mu\nu}}, \quad \boxed{I^{(2)} := \frac{1}{8} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}.$$

[7] Belitz calls them scalars, but I think they are scalar *fields*.

*Remark* 1.

$$\begin{aligned} I^{(1)} \text{ is a scalar field} \quad &\Longrightarrow \quad \tilde{I}^{(1)} = I^{(1)} \text{ in all inertial frames.} \\ I^{(2)} \text{ is a pseudoscalar field} \quad &\Longrightarrow \quad \left|\tilde{I}^{(2)}\right| = \left|I^{(2)}\right| \text{ in all inertial frames.} \end{aligned}$$

The absolute value signs are necessary since $\tilde{I}^{(2)} = (\det D)\, I^{(2)}$.

*Claim* 1. $\boxed{I^{(1)} = \boldsymbol{E}^2 - \boldsymbol{B}^2}$

*Proof.* $I^{(1)} = -\frac{1}{2}\begin{pmatrix} 0 & -\boldsymbol{E} \\ \boldsymbol{E} & B^{jk} \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{E} \\ -\boldsymbol{E} & B_{jk} \end{pmatrix} = \boldsymbol{E}^2 - \frac{1}{2}B^{jk}B_{jk}$. But[a]

$$\frac{1}{2}B^{jk}B_{jk} = \frac{1}{2}\varepsilon_{jkl}B_l\varepsilon_{jkm}B_m = \frac{1}{2}\left(\underbrace{\delta_{kk}}_{3}\delta_{lm} - \underbrace{\delta_{km}\delta_{lk}}_{\delta_{lm}}\right)B_lB_m = \boldsymbol{B}^2$$

$$\Longrightarrow I^{(1)} = \boldsymbol{E}^2 - \boldsymbol{B}^2$$

□

[a] Note that when we write $\boldsymbol{E}^2$, we mean $E_x^2 + E_y^2 + E_z^2$. Also, upper and lower indices don't matter in what follows (Euclidean).

*Claim* 2. $\boxed{I^{(2)} = -\boldsymbol{E}\cdot\boldsymbol{B}}$

*Proof.*

$$\begin{aligned} I^{(2)} &= \frac{1}{8}\left[\varepsilon^{0123}F_{01}F_{23} + \varepsilon^{0132}F_{01}F_{32}\right. \\ &\quad +\varepsilon^{0213}F_{02}F_{13} + \varepsilon^{0231}F_{02}F_{31} \\ &\quad +\varepsilon^{0312}F_{03}F_{12} + \varepsilon^{0321}F_{03}F_{21} \\ &\quad \left. + (4\times 6 = 24 \text{ other terms})\right] \\ &= \frac{1}{4}\left[\varepsilon^{0123}F_{01}F_{23} + \varepsilon^{0213}F_{02}F_{13} + \varepsilon^{0312}F_{03}F_{12}\right. \\ &\quad \left. + (12 \text{ other terms})\right] \\ &= \frac{1}{4}\left[-E_xB_x - E_yB_y - E_zB_z\right]\times 4 = -\boldsymbol{E}\cdot\boldsymbol{B} \end{aligned}$$

□

**Proposition 1.** *The field combinations $I^{(1)}$ and $I^{(2)}$ are invariant under (proper) Lorentz transformations; i.e., their absolute values have the same values in all inertial frames.*

*Proof.* See above. □

*Remark* 2. If $\boldsymbol{E}\perp\boldsymbol{B}$ in some inertial frame, then $\boldsymbol{E}\perp\boldsymbol{B}$ in all other inertial frames

*Remark* 3. Ditto if $\boldsymbol{E}^2 = \boldsymbol{B}^2$ in some frame.

# 5 The superposition principle of Maxwell theory

## 5.1 Real solutions

**Proposition 1.** *Let $\rho^{(\alpha)}(x)$, $\boldsymbol{j}^{(\alpha)}(x)$, with $\alpha = 1, 2$, be two charge and current densities. Let $\boldsymbol{E}^{(\alpha)}(x)$, $\boldsymbol{B}^{(\alpha)}(x)$ be solutions of Maxwell's equations for $\rho^{(\alpha)}$, $\boldsymbol{j}^{(\alpha)}$, and let $\lambda^{(\alpha)} \in \mathbb{R}$. Then*

$$\begin{aligned} \boldsymbol{E} &= \lambda^{(1)}\boldsymbol{E}^{(1)} + \lambda^{(2)}\boldsymbol{E}^{(2)}, \\ \boldsymbol{B} &= \lambda^{(1)}\boldsymbol{B}^{(1)} + \lambda^{(2)}\boldsymbol{B}^{(2)} \end{aligned}$$

*are solutions for*

$$\begin{aligned} \rho &= \lambda^{(1)}\rho^{(1)} + \lambda^{(2)}\rho^{(2)}, \\ \boldsymbol{j} &= \lambda^{(1)}\boldsymbol{j}^{(1)} + \lambda^{(2)}\boldsymbol{j}^{(2)}. \end{aligned}$$

*Proof.* $\nabla \cdot \boldsymbol{E} - 4\pi\rho = \underbrace{\nabla \cdot \boldsymbol{E}^{(1)} - 4\pi\rho^{(1)}}_{=0} + \underbrace{\nabla \cdot \boldsymbol{E}^{(2)} - 4\pi\rho^{(2)}}_{=0} = 0$, etc. □

*Remark* 1. This is obviously true since the theory is linear!

*Remark* 2. If the action contained terms of higher than second order in $F_{\mu\nu}$, this would not be true.

*Remark* 3. A field theory that leads to linear field equations is called *Gaussian* or *free*.

**Corollary 1.** *Let $\boldsymbol{E}^{(k)}(x)$, $\boldsymbol{B}^{(k)}(x)$ be solutions for $\rho^{(k)}(x)$, $\boldsymbol{j}^{(k)}(x)$, where $k \in \mathbb{R}$, and let $\lambda(k) : \mathbb{R} \to \mathbb{R}$ be sufficiently well behaved. Then*

$$\begin{aligned} \boldsymbol{E}(x) &= \int dk\, \lambda(k)\, \boldsymbol{E}^{(k)}(x), \\ \boldsymbol{B}(x) &= \int dk\, \lambda(k)\, \boldsymbol{B}^{(k)}(x) \end{aligned}$$

*are solutions for*

$$\begin{aligned} \rho(x) &= \int dk\, \lambda(k)\, \rho^{(k)}(x), \\ \boldsymbol{j}(x) &= \int dk\, \lambda(k)\, \boldsymbol{j}^{(k)}(x). \end{aligned}$$

*Proof.* Generalize Proposition 1 to $\alpha = 1, \ldots, N$ and let $N \to \infty$. □

*Remark* 4. This can easily be generalized to $\boldsymbol{E}^{(\boldsymbol{k})}(x)$, where $\boldsymbol{k} \in \mathbb{R}^3$.

**Corollary 2.** *The most general solution of Maxwell's equations is obtained as*

$$\begin{aligned} \boldsymbol{E}(x) &= \boldsymbol{E}^{(0)}(x) + \boldsymbol{E}^{(p)}(x), \\ \boldsymbol{B}(x) &= \boldsymbol{B}^{(0)}(x) + \boldsymbol{B}^{(p)}(x) \end{aligned}$$

*where $\boldsymbol{E}^{(0)}$, $\boldsymbol{B}^{(0)}$ are the most general solutions of the homogeneous equations*[a] *and $\boldsymbol{E}^{(p)}$, $\boldsymbol{B}^{(p)}$ is a* *particular solution in the presence of* $\rho$, $\boldsymbol{j}$.

[a] That is, when $\rho = 0$, $\boldsymbol{j} = \mathbf{0}$.

*Proof.* Let $\boldsymbol{E}$, $\boldsymbol{B}$ *be any solution for* $\rho$, $\boldsymbol{j}$, *and let* $\boldsymbol{E}^{(p)}$, $\boldsymbol{B}^{(p)}$ *be a particular solution.*

By Proposition 1,

$$\begin{aligned} \boldsymbol{E}^{(0)} &:= \boldsymbol{E} - \boldsymbol{E}^{(p)}, \\ \boldsymbol{B}^{(0)} &:= \boldsymbol{B} - \boldsymbol{B}^{(p)} \end{aligned}$$

are solutions for $\rho = 0 = \boldsymbol{j}$.

Conversely, if $\boldsymbol{E}^{(0)}$, $\boldsymbol{B}^{(0)}$ is a solution for $\rho = 0 = \boldsymbol{j}$, and $\boldsymbol{E}^{(p)}$, $\boldsymbol{B}^{(p)}$ is some solution for $\rho$, $\boldsymbol{j}$, then

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(p)}, \\ \boldsymbol{B} &= \boldsymbol{B}^{(0)} + \boldsymbol{B}^{(p)} \end{aligned}$$

is a solution for $\rho$, $\boldsymbol{j}$. □

## 5.2 Complex solutions

All physical solutions to Maxwell's equations must consist of real fields $\boldsymbol{E}$, $\boldsymbol{B}$. However, it is sometimes convenient to find complex solutions and take the real part afterwards.

**Proposition 1.** *Let* $\boldsymbol{E}$, $\boldsymbol{B}$ *be complex solutions for complex sources* $\rho$, $\boldsymbol{j}$. *Then* $\boldsymbol{E}^*$, $\boldsymbol{B}^*$ *are solutions for* $\rho^*$, $\boldsymbol{j}^*$.

*Proof.*

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \quad \Longrightarrow \quad \nabla \cdot (\mathrm{Re}\ \boldsymbol{E}) + i\nabla \cdot (\mathrm{Im}\ \boldsymbol{E}) = 4\pi\,(\mathrm{Re}\ \rho) + i4\pi\,(\mathrm{Im}\ \rho)$$

$$\begin{aligned} \Longrightarrow \nabla \cdot (\mathrm{Re}\ \boldsymbol{E}) &= 4\pi\,(\mathrm{Re}\ \rho) \\ \Longrightarrow \nabla \cdot (\mathrm{Im}\ \boldsymbol{E}) &= 4\pi\,(\mathrm{Im}\ \rho) \end{aligned}$$

$$\Longrightarrow \nabla \cdot (\mathrm{Re}\ \boldsymbol{E} - i\mathrm{Im}\ \boldsymbol{E}) = 4\pi\,(\mathrm{Re}\ \rho - i\mathrm{Im}\ \rho) \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{E}^* = 4\pi\rho^*,$$

etc. for the other Maxwell equations. □

*Remark* 1. This, again, is because of linearity.

**Corollary 1.** *Let* $\boldsymbol{E}$, $\boldsymbol{B}$ *be complex solutions for real (i.e. physical) sources* $\rho$, $\boldsymbol{j}$. *Then* Re $\boldsymbol{E}$, Re $\boldsymbol{B}$ *are also solutions for* $\rho$, $\boldsymbol{j}$.

*Proof.* From Corollary (?), Re $\boldsymbol{E}$, Re $\boldsymbol{B}$ are solutions for Re $\rho = \rho$, Re $\boldsymbol{j} = \boldsymbol{j}$. □

*Remark* 2. In this case, Im $\boldsymbol{E}$, Im $\boldsymbol{B}$ are solutions in the absence of sources (since Im $\rho = 0$, Im $\boldsymbol{j} = \mathbf{0}$).

# Chapter 3

# Static solutions of Maxwell's equations

## 1 Poisson's equations

### 1.1 Electrostatics

Consider Maxwell's equations for static fields:

$$\begin{aligned} (M2) &\quad\rightarrow\quad \nabla \times \boldsymbol{E} = 0 \\ (M3) &\quad\rightarrow\quad \nabla \cdot \boldsymbol{E} = 4\pi\rho. \end{aligned}$$

$$\begin{aligned} (M1) &\quad\rightarrow\quad \nabla \cdot \boldsymbol{B} = 0 \\ (M4) &\quad\rightarrow\quad \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} \end{aligned}$$

*Remark* 1. $(M2)$ and $(M3)$ now contain $\boldsymbol{E}$ only! $(M1)$ and $(M4)$ now contain $\boldsymbol{B}$ only! For static fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ decouple!

*Remark* 2. From Ch. 2 § 3.4, a static $\boldsymbol{E}-$field is determined by $\phi$ alone:

$$\boxed{\boldsymbol{E} = -\nabla\phi},$$

and $(M2)$ is thus automatically satisfied, since $(\nabla \times \boldsymbol{E})_i = -(\nabla \times \nabla\phi)_i = 0$.

**Proposition 1.** *The electrostatic potential $\phi(\boldsymbol{x})$ obeys* Poisson's equations *for $\rho(\boldsymbol{x})$:*

$$\boxed{\nabla^2\phi = -4\pi\rho},$$

*where $\nabla^2 := \partial_j\partial^j =: \Delta$ is the* Laplace operator.

**Corollary 1.** *In vacuum, $\phi(\boldsymbol{x})$ obeys the* Laplace equation*:*

$$\boxed{\nabla^2\phi = 0}.$$

*Remark* 3. Solutions of Laplace's equation are called *harmonic functions*.

*Remark* 4. $\phi(\boldsymbol{x}) = \text{const.}$, $x$, $y$, $z^2 - \frac{1}{2}\left(x^2 + y^2\right)$ are all harmonic functions.

*Remark* 5. A harmonic function can have no extrema except at infinity; this is a theorem in analysis.

### 1.2 Magnetostatics

From Ch. 2, § 3.4, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

*Remark* 1. This is always true!

*Remark* 2. $(M1)$ is automatically fulfilled, since $\nabla \cdot \nabla \times \boldsymbol{A} = 0$.

**Proposition 1.** *The static Euclidean vector potential $\boldsymbol{A}(\boldsymbol{x})$ obeys*

$$\boxed{\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}},$$

*where $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x})$.*

*Proof.*

$$\begin{aligned}(\nabla \times \nabla \times \boldsymbol{A})_i &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\partial_l A_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l A_m \\ &= -\partial_j\partial_j A_i + \partial_i\partial_j A_j \\ &= -\nabla^2 A_i + \partial_i (\nabla \cdot \boldsymbol{A}).\end{aligned}$$

But by Problem #19, we can always choose the Coulumb gauge to make $\nabla \cdot \boldsymbol{A} = 0$.

$$\Longrightarrow \frac{4\pi}{c}\boldsymbol{j} = \nabla \times \boldsymbol{B} = \nabla \times \nabla \times \boldsymbol{A} = -\nabla^2 \boldsymbol{A}.$$

□

*Remark* 3. Combining Propositions 1 and 1, we see that the components of the static electromagnetic potential obey Poisson's equation with $-\frac{4\pi}{c}$ times the components $(c\rho, \boldsymbol{j})$ of the 4-current as the inhomogeneity.

*Remark* 4. Poisson's equation is linear; thus, the most general solution is a particular solution plus the most general solution of Laplace's equation (see Ch. 2 § 5.1 Corollary 2).

*Remark* 5. From § 1.1 Remark 5, the only solution of Laplace's equation that vanishes at infinity is the zero solution; in an infinite system, there is only one physical solution of Poisson's equation.

*Remark* 6. Things get more complicated in a finite system with boundary conditions.

## 2 Digression: Fourier transforms and generalized functions

### 2.1 The Fourier transform in classical analysis

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex-valued function of $n$ real arguments that is *absolutely integrable*:

$$\int d\boldsymbol{x}\ |f(\boldsymbol{x})| < \infty.$$

*Remark* 1. The space of these functions, denoted $\gamma^{(1)}$, forms a vector space over $\mathbb{C}$ under addition of functions.

**Notation:**
$\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,
$\int d\boldsymbol{x} = \int_{\mathbb{R}^n} dx_1 \cdots dx_n$,
$\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + \cdots + k_n x_n \quad (\boldsymbol{k} \in \mathbb{R}^n)$.

**Definition 1.** ***Fourier transform.*** The *Fourier transform* of $f(\boldsymbol{x})$ is defined as

$$\boxed{\hat{f}(\boldsymbol{k}) := \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) =: \mathcal{F}[f(\boldsymbol{x})](\boldsymbol{k})}$$

*Remark* 2. $\hat{f}:\mathbb{R}^n \to \mathbb{C}$ is another complex-valued function of $\mathbb{R}^n$.

*Remark* 3. The Fourier transform is a *linear integral transform.*

*Remark* 4. $\mathcal{F}[\lambda_1 f_1 + \lambda_2 f_2] = \lambda_1 \mathcal{F}[f_1] + \lambda_2 \mathcal{F}[f_2] \quad \forall \lambda_{1,2} \in \mathbb{C}$ due to this linearity.

**Proposition 1.** $\hat{f}(\boldsymbol{k})$ *is* bounded *and* continuous.

*Proof.* To show that $\hat{f}$ is bounded,

$$\left|\hat{f}(\boldsymbol{k})\right| = \left|\int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\right| \leq \int d\boldsymbol{x}\, \left|e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\right| = \int d\boldsymbol{x}\, |f(\boldsymbol{x})| < \infty,$$

where we have used the triangle inequality.

To show that $\hat{f}$ is continuous,

$$\begin{aligned} \left|\hat{f}(\boldsymbol{k}_1) - \hat{f}(\boldsymbol{k}_2)\right| &= \left|\int d\boldsymbol{x}\, \left(e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}}\right) f(\boldsymbol{x})\right| \leq \int d\boldsymbol{x}\, \left|e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}}\right| |f(\boldsymbol{x})| \\ &\to 0 \quad \text{for} \quad \boldsymbol{k}_1 \to \boldsymbol{k}_2, \end{aligned}$$

where, again, we have used the triangle inequality. □

**Proposition 2.** *Let $x_l f(\boldsymbol{x})$ be absolutely integrable. Then $\hat{f}(\boldsymbol{k})$ is differentiable with respect to $k_l$ and*

$$\boxed{\frac{\partial}{\partial k_l}\hat{f}(\boldsymbol{k}) = \mathcal{F}[-ix_l f](\boldsymbol{k})}.$$

*Proof.* $\frac{\partial}{\partial k_l}\hat{f}(\boldsymbol{k}) = \frac{\partial}{\partial k_l}\int d\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}) = -i\int d\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} x_l f(\boldsymbol{x}) = \mathcal{F}[-ix_l f](\boldsymbol{k})$. Note that we needed to stipulate that $x_l f(\boldsymbol{x})$ was absolutely integrable to proceed with the last step. □

**Proposition 3.** *Let $f(\boldsymbol{x})$ be differentiable with respect to $x_l$, and let $\frac{\partial}{\partial x_l} f$ be absolutely integrable. Then*

$$\boxed{\mathcal{F}[\partial_l f](\boldsymbol{k}) = ik_l \hat{f}(\boldsymbol{k})}.$$

*Proof.* For $n = 1$,

$$\int dx\, e^{-ikx}\frac{d}{dx} f(x) = \underbrace{e^{-ikx} f(x)\big|_{-\infty}^{\infty}}_{=0} - \int dx\, (-ik)\, e^{-ikx} f(x) = ik\hat{f}(k)\,.$$

□

*Remark* 5. The Fourier transform turns derivatives into products! Prospect: turn differential equations into algebraic ones!

*Remark* 6. This also works for $n > 1$ and higher derivatives. For instance, for $n = 3$,

$$\boxed{\mathcal{F}\left[\nabla^2 f\right](\boldsymbol{k}) = -\boldsymbol{k}^2 \hat{f}(\boldsymbol{k})}.$$

**Proposition 4.**

$$\boxed{\mathcal{F}[f^*](\boldsymbol{k}) = (\mathcal{F}[f](-\boldsymbol{k}))^*}.$$

*Proof.* $\mathcal{F}[f^*](\boldsymbol{k}) = \int d\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f^*(\boldsymbol{x}) = \left(\int d\boldsymbol{x} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\right)^* = \left(\hat{f}(-\boldsymbol{k})\right)^*.$ □

**Theorem 1. Convolution theorem.** *Let $f_1$, $f_2$ be absolutely integrable, and let their* convolution *$f_1 \star f_2$, defined as*

$$\boxed{(f_1 \star f_2)(\boldsymbol{y}) := \int d\boldsymbol{x}\, f_1(\boldsymbol{y}-\boldsymbol{x})\, f_2(\boldsymbol{x})},$$

*exist and be absolutely integrable. Then*

$$\boxed{\mathcal{F}[f_1 \star f_2](\boldsymbol{k}) = \hat{f}_1(\boldsymbol{k})\, \hat{f}_2(\boldsymbol{k})}.$$

*Proof.*

$$\begin{aligned}
\mathcal{F}[f_1 \star f_2](\boldsymbol{k}) &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \int d\boldsymbol{x}\, f_1(\boldsymbol{y}-\boldsymbol{x})\, f_2(\boldsymbol{x}) \\
&= \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})} f_1(\boldsymbol{y}-\boldsymbol{x})\, f_2(\boldsymbol{x}) \\
&= \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f_2(\boldsymbol{x}) \int d\boldsymbol{z} e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} f_1(\boldsymbol{z})
\end{aligned}$$

□

*Remark* 7. Convolutions in real space turn into products in Fourier space.

## 2.2 Inverse Fourier transforms

Let $f_1$, $f_2$ be absolutely integrable.

**Lemma 1.** $\int d\boldsymbol{x}\, f_1(\boldsymbol{x}) \left(\hat{f}_2(\boldsymbol{x})\right)^* = \int d\boldsymbol{y}\, \hat{f}_1(-\boldsymbol{y})\,(f_2(\boldsymbol{y}))^*.$

*Proof.*

$$\begin{aligned}\int d\boldsymbol{x}\, f_1(\boldsymbol{x})\left(\hat{f}_2(\boldsymbol{x})\right)^* &= \int d\boldsymbol{x}\, f_1(\boldsymbol{x})\left(\int d\boldsymbol{y}\, e^{-i\boldsymbol{x}\cdot\boldsymbol{y}} f_2(\boldsymbol{y})\right)^* \\ &= \int d\boldsymbol{x}\, f_1(\boldsymbol{x})\int d\boldsymbol{y}\, e^{i\boldsymbol{x}\cdot\boldsymbol{y}}\left(f_2(\boldsymbol{y})\right)^* \\ &= \int d\boldsymbol{y}\, \left(f_2(\boldsymbol{y})\right)^* \underbrace{\int d\boldsymbol{x}\, f_1(\boldsymbol{x})\, e^{i\boldsymbol{x}\cdot\boldsymbol{y}}}_{\hat{f}_1(-\boldsymbol{y})} \\ &= \int d\boldsymbol{y}\, \hat{f}_1(-\boldsymbol{y})\left(f_2(\boldsymbol{y})\right)^*\end{aligned}$$

□

**Theorem 1. Inverse Fourier transform.** *Let $f(\boldsymbol{x})$ and $\hat{f}(\boldsymbol{k})$ exist and be absolutely integrable. Then the inverse Fourier transform is*

$$\boxed{f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{f}(\boldsymbol{k}) =: \mathcal{F}^{-1}\left[\hat{f}\right](\boldsymbol{x})}.$$

*Remark* 1. This means $\mathcal{F}[\mathcal{F}[f]] = (2\pi)^n f$; i.e., the Fourier transform is its own inverse apart from a factor of $(2\pi)^n$.

*Proof.* Consider Lemma 1 with $f_1 = f$, $f_2(\boldsymbol{y}) = e^{-\alpha\boldsymbol{y}^2}e^{i\boldsymbol{y}\cdot\boldsymbol{x}}$, where $\alpha > 0$.

$$\begin{aligned}\Longrightarrow \hat{f}_2(\boldsymbol{k}) &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}e^{-\alpha\boldsymbol{y}^2}e^{i\boldsymbol{x}\cdot\boldsymbol{y}} \\ &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{y}\cdot(\boldsymbol{k}-\boldsymbol{x})}e^{-\alpha\boldsymbol{y}^2} \\ &= \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{1}{4\alpha}(\boldsymbol{k}-\boldsymbol{x})^2},\end{aligned}$$

where the last step, that Fourier transform of a Gaussian is a Gaussian, is the result of Problem #27. By the Lemma,

$$\int d\boldsymbol{y}\, \hat{f}(-\boldsymbol{y})\underbrace{e^{-\alpha\boldsymbol{y}^2}e^{-i\boldsymbol{y}\cdot\boldsymbol{x}}}_{(f_2(\boldsymbol{y}))^*} = \int d\boldsymbol{k}\, f(\boldsymbol{k})\underbrace{\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{1}{4\alpha}(\boldsymbol{x}-\boldsymbol{k})^2}}_{\left(\hat{f}_2(\boldsymbol{k})\right)^*}.$$

Consider the limit as $\alpha \to 0$.

On the left hand side,

$$\lim_{\alpha\to 0}\int d\boldsymbol{y}\, \hat{f}(-\boldsymbol{y})\, e^{-\alpha\boldsymbol{y}^2}e^{-i\boldsymbol{y}\cdot\boldsymbol{x}} = \underline{\int d\boldsymbol{k}\, \hat{f}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}$$

On the right hand side, by the Intermediate Value Theorem,

$$\begin{aligned}
\lim_{\alpha\to 0}\int d\boldsymbol{k}\, f(\boldsymbol{k})\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{1}{4\alpha}(\boldsymbol{x}-\boldsymbol{k})^2} &= f(\boldsymbol{x})\lim_{\alpha\to 0}\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}\int d\boldsymbol{k}\, e^{-\frac{1}{4\alpha}(\boldsymbol{x}-\boldsymbol{k})^2}\\
&= f(\boldsymbol{x})\lim_{\alpha\to 0}\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}\left(\int d\boldsymbol{k}\, e^{-\frac{1}{4\alpha}\boldsymbol{k}^2}\right)^n\\
&= f(\boldsymbol{x})\lim_{\alpha\to 0}\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}\left(2\sqrt{a}\right)^n\left(\sqrt{\pi}\right)^n\\
&= \underline{(2\pi)^n f(\boldsymbol{x})}
\end{aligned}$$

□

## 2.3 Test functions

**problem:** In classical analysis, very few functions are Fourier transformable, and even simple functions are not Fourier transformable

**solution:** "Generalized functions" (sometimes called "distributions")

In order to define generalized functions, we first consider function spaces in addition to $\gamma^{(1)}$.

**Definition 1.** ***Test functions.*** A function $F:\mathbb{R}\to\mathbb{C}$ is called a *test function* iff

(i) $F$ is differentiable arbitrarily many times, and

(ii) $F$ and all of its derivatives go to zero faster than any power[a] $|x|\to\infty$.

[a] That is, $\lim_{x\to\infty} x^N F^{(n)}(x) = 0$ for all $N, n\in\mathbb{N}$.

**Example 1.** $F(x) = e^{-x^2}$ is a test function, So is $x^n e^{-mx^2}$ for all $m,n\in\mathbb{N}$.

**Definition 2.** ***Weakly increasing functions.*** A function $\phi:\mathbb{R}\to\mathbb{C}$ is called a *weakly increasing function* iff

(i) $\phi$ is differentiable arbitrarily many times, and

(ii) $\phi$ and all its derivatives do not grow faster than $|x|^N$ for $|x|\to\infty$, where $N\in\mathbb{N}$ may depend on the order of the derivative.

**Example 2.** Any polynomial is a weakly increasing function, but $e^x$ is not.

*Remark* 1. The derivative of a test function is a test function; so is the sum of two test functions, as well as scalar multiples of test functions. **Thus, the set of test functions forms a vector space; we call it $\gamma$.**

*Remark* 2. Let $F$ be a test function and let $\phi$ be a weakly increasing function. Then

$$G(x) := F(x)\,\phi(x)$$

is a test function.

Now, test functions are all Fourier transformable since they are absolute-integrable (they die off at infinitely very fast). But is the Fourier transform of a test function also a test function?

**Proposition 1.** *If $F(x)$ is a test function, then so is its Fourier transform, $\hat{F}(k) := \int dx e^{-ikx} F(x)$.*

*Proof.* Consider the $p^{\text{th}}$ derivative of $\hat{F}(k)$:

$$\hat{F}^{(p)}(k) := \frac{d^p}{dk^p}\hat{F}(k) = (-i)^p \int dx\, x^p F(x)\, e^{-ikx} = (-i)^p \mathcal{F}\left[x^p F(x)\right](k)$$

By the two remarks above, $x^p F(x)$ is a test function, and so $\mathcal{F}\left[x^p F(x)\right](k)$ exists.

$$\begin{aligned} \left|\hat{F}^{(p)}(k)\right| &= \left|\int dx\, x^p F(x)\, e^{-ikx}\right| \\ &= \left|\int dx\, x^p F(x)\, \frac{1}{-ik}\frac{d}{dx}e^{-ikx}\right| \end{aligned}$$

Integrating by parts (the boundary term vanishes since $F$ is a test function):

$$\Longrightarrow \left|\hat{F}^{(p)}(k)\right| = \left|\frac{1}{-ik}\int dx\, e^{-ikx}\frac{d}{dx}\left(x^p F(x)\right)\right|.$$

We can do this again to pile on more derivatives onto $x^p F(x)$ at the cost of a term $\frac{1}{-ik}$. Doing this $N-1$ more times (where $N \in \mathbb{N}$ is arbitrary), we get

$$\left|\hat{F}^{(p)}(k)\right| = \left|\left(\frac{1}{-ik}\right)^N \int dx\, e^{-ikx}\frac{d^N}{dx^N}\left(x^p F(x)\right)\right|.$$

By the triangle inequality, this becomes

$$\left|\hat{F}^{(p)}(k)\right| \leq \frac{1}{|k|^N}\underbrace{\int dx \left|\frac{d^N}{dx^N}\left(x^p F(x)\right)\right|}_{<\infty \text{ since } F\in\gamma} = O\left(\frac{1}{|k|^N}\right).$$

Since $N$ can be made arbitrarily large, $\left|\hat{F}^{(p)}(k)\right|$ falls off faster than any power. Thus, $\hat{F}^{(p)}(k)$ is a test function.[a] □

[a] Specifically, $\hat{F}^{(0)}(k) = \hat{F}(k)$ is a test function, so the proposition is true.

*Remark* 3. The inverse Fourier transform is given by the theorem in § 2.2.

**Proposition 2. Parseval's equation.** *Let $F_1(x)$ and $F_2(x)$ be test functions, and let $\hat{F}_1(k)$ and $\hat{F}_2(k)$ be their Fourier transforms. Then*

$$\int \frac{dk}{2\pi}\,\hat{F}_1(k)\,\hat{F}_2(k) = \int dx\, F_1(x)\, F_2(-x).$$

*Proof.* $\int \frac{dk}{2\pi}\hat{F}_1(k)\,\hat{F}_2(k) = \int \frac{dk}{2\pi}\int dx e^{-ikx} F_1(x)\,\hat{F}_2(k) = \int dx F_1(x)\int \frac{dk}{2\pi}\hat{F}_2(k)\, e^{-ikx} = \int dx F_1(x)\, F_2(-x)$. □

## 2.4 Generalized functions

**Definition 1.** ***Regular sequences.*** Let $n \in \mathbb{N}$, and let $\{f_n(x)\}$ be a sequence of test functions. The sequence is called *regular* iff

$$\lim_{n\to\infty}\int dx\, f_n(x)\, F(x) \tag{3.1}$$

exists for all test functions $F(x)$.

*Remark* 1. The integral exists for all $n$, so the only issue is whether the limit exists.

**Example 1.** Consider the sequence $\left\{e^{-\frac{x^2}{n^2}}\right\}$, where $n \in \mathbb{N}$. This sequence is regular since $\lim_{n\to\infty}\int dx e^{-\frac{x^2}{n^2}} F(x) = \int dx F(x)$ for all $F \in \gamma$. For proof, see Problem #30.

**Definition 2.** ***Equivalence of regular sequences.*** Two regular sequences of test functions are called *equivalent* iff their limits from Equation (3.1) are equal.

**Example 2.** $\left\{e^{-\frac{x^2}{n^4}}\right\}$ is equivalent to $\left\{e^{-\frac{x^2}{n^2}}\right\}$; so is $\left\{e^{-\frac{x^2}{n}}\right\}$.

**Definition 3.** ***Generalized functions, regularizations.*** The set of all equivalent regular sequences $\{f_n(x)\}$ defines a *generalized function* (or *distribution*)[a] $f(x)$, and we define the integral

$$\int dx\, f(x)\, F(x) := \lim_{n\to\infty}\int dx\, f_n(x)\, F(x)$$

by the limit on the right hand side, which exists for all $F \in \gamma$ and is the same for all of the equivalent sequences.

Any of the equivalent sequences is called a *regularization* of the generalized function $f(x)$.

[a] In other words, if we let $L \in \mathbb{C}$, then $f(x) = \left\{\{f_n(x)\} : \lim_{n\to\infty}\int dx\, f_n(x)\, F(x) = L\right\}$. (Note to reader: I am not sure if this is correct, gotta double check)

**Example 3.** $\left\{e^{-\frac{x^2}{n^2}}\right\}$ and its equivalent sequences define the generalized function $f(x) = 1$. $\left\{e^{-\frac{x^2}{n^2}}\right\}$ is a regularization of $f(x) = 1$.

*Remark* 2. The properties of the generalized function $f(x) = 1$ coincide with those of the ordinary function.

*Remark* 3. Differentiation, addition, multiplication with weakly increasing functions, and Fourier transforms of generalized functions can all be defined in terms of their regularizations; doing so yields generalized functions. However, multiplication between two generalized functions can *not* be consistently defined.

**Proposition 1.** *Let $f(x)$ be a function (in the ordinary sense) such that there exists an $N \in \mathbb{N}$ such that $\frac{f(x)}{(1+x^2)^N}$ is absolutely integrable.*

*Then one can construct sequences of test functions $\{f_n(x)\}$ such that $\lim_{n\to\infty}\int dx\, f_n(x)\, F(x) = \int dx\, f(x)\, F(x)$ for all test functions $F(x)$.*

*Proof.* See books (e.g., Lighthill Chapter 2.3). ☐

*Remark* 4. This result says that a large class of ordinary functions can be considered generalized functions.

**Example 4.** Consider the ordinary function sgn $x := \frac{|x|}{x}$. This function fulfills the premise of Proposition 1 for $N = 1$. Thus, sgn $x$ is a generalized function. One regularization is $\{\tanh(nx)\}$ (for proof see Problem #31).

*Remark* 5. Such constructed generalized functions are called *regular generalized functions.* The derivative of any regular generalized function exists, but in general it is not regular.

**Example 5.** $\frac{d}{dx}\text{sgn } x$ exists as a generalized function, but it is not regular (see Problem #32).

**Definition 4.** ***Distribution limit.*** Let $f_t(x)$ be a generalized function for any value of the parameter $t$, and let $f(x)$ be another generalized function such that

$$\lim_{t\to c}\int dx\, f_t(x)\, F(x) = \int dx\, f(x)\, F(x)$$

for all test functions $F(x)$, where $c$ may be finite or infinite, and the set of parameters $t$ may be continuous or discrete. Then we say

$$\lim_{t\to c} f_t(x) = f(x)\,.$$

*Remark* 6. This is sometimes called a *distribution limit.*

**Example 6.** $\lim_{\epsilon\to 0}|x|^{\epsilon}\,\text{sgn } x = \text{sgn } x$. See Problem #31(c) for more.

**Example 7.** Consider the test functions $f_n(x)$ that make up a regular sequence (in the sense of Definition 1) to be generalized functions (math books say we can), and let $f(x)$ be the generalized function that is defined by this sequence and its equivalence class. Then

$$\lim_{n\to\infty} f_n(x) = f(x)\,.$$

**Proposition 2.** *Under the conditions of Definition 4, we have*

*(i)* $\lim_{t\to c} f_t'(x) = f'(x)$

*(ii)* $\lim_{t\to c} f_t(ax+b) = f(ax+b)$

*(iii)* $\lim_{t\to c}\phi(x)\, f_t(x) = \phi(x)\, f(x)$ *for any weakly increasing function* $\phi(x)$.

*Proof.* Math books. □

## 2.5 The $\delta-$function

**Definition 1.** ***Dirac delta function.*** The generalized function $\delta(x)$ is defined as the set of equivalent regular sequences (of test functions) for which

$$\boxed{\int dx\,\delta(x)\, F(x) = \lim_{n\to\infty}\int dx\, f_n(x)\, F(x) = F(0) \quad \forall F\in\gamma}\,.$$

*Remark* 1. There is no ordinary function that has this property.

**Proposition 1.** *One regularization of $\delta(x)$ is the sequence defined by*

$$\boxed{f_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2} \qquad (n \in \mathbb{N}).}$$

*Proof.* First, note that $f_n$ are test functions, as required by Definition 3. Also note that

$$\int dx\, f_n(x) = \sqrt{\frac{n}{\pi}} \int dx\, e^{-nx^2} = \frac{1}{\sqrt{\pi}} \int dx\, e^{-x^2} = 1.$$

$$\begin{aligned}
\Longrightarrow \left| \int dx\, f_n(x) F(x) - F(0) \right| &= \left| \int dx\, f_n(x) (F(x) - F(0)) \right| \\
\leq \int dx\, f_n(x) |F(x) - F(0)| &= \int dx\, f_n(x) |x| \left| \frac{F(x) - F(0)}{x} \right| \\
\leq (\sup F') \int dx\, |x| f_n(x) &= 2 (\sup F') \int_0^\infty dx\, x \sqrt{\frac{n}{\pi}} e^{-nx^2} \\
&= \frac{2}{\sqrt{n\pi}} \underbrace{(\sup F')}_{\text{const.}} \underbrace{\int_0^\infty dx\, x e^{-x^2}}_{\text{indep. of } n} \\
&\to 0 \quad \text{for} \quad n \to \infty,
\end{aligned}$$

where the first inequality is the triangle inequality, and the second inequality comes from the fact that $F'$ is bounded, allowing us to pull out the (finite) supremum of $F'$. □

**Proposition 2.** *The Fourier transform of $\delta(x)$ is*

$$\boxed{\hat{\delta}(k) = 1}.$$

*Proof.* Consider the regularization $f_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2}$. From Problem #27, $\hat{f}_n(k) = e^{-\frac{k^2}{4n}}$. But from § 2.4 Example 3, this is a regularization of the generalized function that is identically equal to 1. □

**Corollary 1.** *The $\delta-$function can be written*

$$\boxed{\delta(x) = \int \frac{dk}{2\pi} e^{ikx}}.$$

*Proof.* From the theorem from § 2.2,

$$\delta(x) = \int \frac{dk}{2\pi} e^{ikx} \hat{\delta}(k) = \int \frac{dk}{2\pi} e^{ikx}.$$

□

*Remark* 2. This integral does not exist in classical analysis!

**Proposition 3.** *Let $\phi(x)$ be a weakly increasing function.*[a] *Then*

$$\phi(x)\,\delta(x) = \phi(0)\,\delta(x)\,.$$

[a] As per § 2.3 Definition 2.

*Proof.* $\int dx\,\delta(x)\underbrace{\phi(x)\,F(x)}_{\text{test fct.}} = \phi(0)\,F(0) = \phi(0)\int dx\,\delta(x)\,F(x) \quad \forall F \in \gamma..$ □

**Corollary 2.** *Let $\phi(x)$ be a weakly increasing function. Then*[a]

$$\boxed{\int dx\,\delta(x)\,\phi(x) = \phi(0)}.$$

[a] This result says we can now use the $\delta-$function with weakly increasing functions!

*Proof.* $\int dx\,\delta(x)\,\phi(x) = \phi(0)\int dx\,\delta(x) = \phi(0)\,\hat{\delta}(k=0) = \phi(0)\,.$ □

*Remark* 3. This is consistent with $\hat{\delta}(k) = \int dx\,e^{-ikx}\delta(x) = 1$.

*Remark* 4. We can define *even* and *odd* generalized functions in analogy to the definitions for ordinary functions:

**Example 1.** $\delta(x) = \delta(-x)$ is even, since $\delta(-x) = \int \frac{dk}{2\pi}\,e^{-ikx} = \int \frac{dk}{2\pi}\,e^{ikx} = \delta(x)\,.$ Accordingly, $\delta'(x) := \frac{d}{dx}\delta(x)$ is odd.

*Remark* 5. The $\delta-$function makes Fourier back transforms easy:

$$\int \frac{dk}{2\pi}\,\hat{f}(k)\,e^{ikx} = \int \frac{dk}{2\pi}\,e^{ikx}\int dy\,e^{-iky}f(y) = \int dy\,\delta(y-x)\,f(y) = f(x)\,.$$

We can now Fourier transform weakly increasing functions, not just absolutely integrable ones!

**Proposition 4.** *The $\delta-$function has the properties*

$$\text{(i)}\quad \delta(ax) = \frac{1}{|a|}\delta(x) \quad \forall a \in \mathbb{R} - \{0\}\,,$$

$$\text{(ii)}\quad f(x)\,\delta(a-x) = f(a)\,\delta(a-x)\,,$$

$$\text{(iii)}\quad \delta(f(x)) = \sum_j \frac{1}{|f'(x_j)|}\delta(x-x_j)\,,$$

*where the $x_j$ are all real zeros of $f(x)$ and we assume they are simple and isolated.*

*Proof.*

(i)
$$\int dx\, F(x)\,\delta(ax) = \text{sgn } a \int \frac{dx}{a}\, F\left(\frac{x}{a}\right)\delta(x) = \frac{\text{sgn } a}{a} F(0) = \frac{1}{|a|} F(0) = \frac{1}{|a|}\int dx\, F(x)\,\delta(x) \quad \forall F \in \gamma.$$

(ii)
$$\int dx\, F(x)\, f(x)\,\delta(x-a) = \int dx\, F(x+a)\, f(x+a)\,\delta(x) = F(a)\, f(a) = \int dx\, F(x)\, f(a)\,\delta(a-x) \quad \forall F \in \gamma.$$

(iii) Let $f(x) =: y, \implies x = f^{-1}(y), dy = f'(x)\,dx$. Then

$$\int dx\, F(x)\,\delta(f(x)) = \sum_j \int_{x_j-\varepsilon}^{x_j+\varepsilon} dx\, F(x)\,\delta(f(x)) = \int_{f(x_j-\varepsilon)}^{f(x_j+\varepsilon)} dy\, \frac{F\left(x = f^{-1}(y)\right)}{\left|f'\left(x = f^{-1}(y)\right)\right|}\delta(y) = \frac{F(x_j)}{|f'(x_j)|} = \sum_j \int dx\, F(x)\, \frac{\delta(x-x_j)}{|f'(x_j)|} \quad \forall F \in \gamma$$

□

**Example 2.** $\delta\left(x^2-a^2\right) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right].$

# 3 Solutions of Poisson's Equation

## 3.1 The general solution of Poisson's equation

**Proposition 1.** *Every Fourier transformable solution of Poisson's equation is uniquely determined by the inhomogeneity $\rho$ via*
$$\phi(\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{4\pi}{\boldsymbol{k}^2}\hat{\rho}(\boldsymbol{k})$$

*Proof.* From § 1.1,[a]

$$\nabla^2\phi = -4\pi\rho \xrightarrow{\mathcal{F}} -\boldsymbol{k}^2\hat{\phi}(\boldsymbol{k}) = -4\pi\hat{\rho}(\boldsymbol{k})$$
$$\implies \hat{\phi}(\boldsymbol{k}) = \frac{4\pi}{\boldsymbol{k}^2}\hat{\rho}(\boldsymbol{k}) \xrightarrow{\mathcal{F}^{-1}} \phi(\boldsymbol{x}) = \mathcal{F}^{-1}\left[\frac{4\pi}{\boldsymbol{k}^2}\hat{\rho}(\boldsymbol{k})\right](\boldsymbol{x}).$$

□

[a] Here and elsewhere, the symbol $\xrightarrow{\mathcal{F}}$ is used to indicate a Fourier transform is taken.

*Remark* 1. Thanks to the theory in § 2, the class of solutions that can be constructed in this way is much larger than before, since weakly increasing functions are allowed.

*Remark* 2. § 1.2 Remark 5 follows immediately:

$$\begin{aligned}\nabla^2\phi = 0 \quad &\iff \quad \boldsymbol{k}^2\hat{\phi}(\boldsymbol{k}) = 0 \\ &\iff \quad \hat{\phi}(\boldsymbol{k}) = 0 \quad \forall \boldsymbol{k} \neq \boldsymbol{0} \\ &\iff \quad \phi(\boldsymbol{x}) = \text{const.}\end{aligned}$$

*Remark* 3. All of this is consistent with § 1.2 Remark 4.

## 3.2 The Coulomb potential

What is the potential from one charge?

Consider a point charge: $\rho(\boldsymbol{x}) = e\delta(\boldsymbol{x})$, where $\delta(\boldsymbol{x}) := \delta(x)\,\delta(y)\,\delta(z)$.

**Theorem 1.** *The electrostatic potential resulting from a point charge is the* Coulomb potential:[a]

$$\boxed{\phi(\boldsymbol{x}) = \frac{e}{r}}.$$

[a] $r := |\boldsymbol{x}|$

*Proof.*

$$\hat{\rho}(\boldsymbol{k}) = \mathcal{F}[e\delta(\boldsymbol{x})](\boldsymbol{k}) = e$$

$$\implies \hat{\phi}(\boldsymbol{k}) = \frac{4\pi}{\boldsymbol{k}^2}e \quad \xrightarrow{\mathcal{F}^{-1}} \quad \phi(\boldsymbol{x}) = \frac{e}{r}$$

(For derivation of this inverse Fourier transform, see Problem #28). □

*Remark* 1. We have now *derived* the Coulomb potential from a least action principle, whereas it was postulated in PHYS 611.

**Corollary 1.** *The electric field of a point charge is*

$$\boxed{\boldsymbol{E}(\boldsymbol{x}) = e\frac{\boldsymbol{x}}{r^3}}.$$

*Proof.* $\boldsymbol{E} = -\nabla\phi = -e\nabla\frac{1}{r} = -e\left(-\frac{1}{2}\frac{2\boldsymbol{x}}{r^3}\right) = e\frac{\boldsymbol{x}}{r^3}$. □

*Remark* 2. The electric field of a point charge is purely radial and isotropic.

## 3.3 Poisson's formula

**Proposition 1.** *Let* $\rho(\boldsymbol{x})$ *be a charge distribution whose Fourier transform exists. Then*

$$\boxed{\phi(\boldsymbol{x}) = \int d\boldsymbol{y}\, \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}}.$$

*This is known as* Poisson's formula.

*Proof.* From § 3.1,

$$\phi(\boldsymbol{x}) = \mathcal{F}^{-1}\left[\underbrace{\frac{4\pi}{\boldsymbol{k}^2}\hat{\rho}(\boldsymbol{k})}_{\hat{\phi}(\boldsymbol{k})}\right](\boldsymbol{x}).$$

We know that

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{4\pi}{\boldsymbol{k}^2}\right](\boldsymbol{x}) &= \frac{1}{|\boldsymbol{x}|},\\ \mathcal{F}^{-1}\left[\hat{\rho}(\boldsymbol{k})\right](\boldsymbol{x}) &= \rho(\boldsymbol{x}).\end{aligned}$$

From the convolution theorem from § 2.1,

$$\begin{aligned}\phi(\boldsymbol{x}) &= \left(\mathcal{F}^{-1}\left[\frac{4\pi}{\boldsymbol{k}^2}\right]\star\mathcal{F}^{-1}\left[\hat{\rho}(\boldsymbol{k})\right]\right)(\boldsymbol{x})\\ &= \int d\boldsymbol{y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho(\boldsymbol{y}).\end{aligned}$$

□

*Remark* 1. For $\rho(\boldsymbol{y}) = e\delta(\boldsymbol{y})$, we get

$$\phi(\boldsymbol{x}) = \int d\boldsymbol{y}\,\underbrace{\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}_{\substack{\text{weakly}\\ \text{inc.}}}\,e\delta(\boldsymbol{y}) = \frac{e}{|\boldsymbol{x}|}.$$

*Remark* 2. $\nabla_{\boldsymbol{x}}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) = -\frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}$

$$\Longrightarrow \boxed{\boldsymbol{E}(\boldsymbol{x}) = \int d\boldsymbol{y}\,\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\rho(\boldsymbol{y})}.$$

## 3.4 The field of a uniformly moving charge

Consider a charge $e$ that moves with constant velocity with respect to an observer. Finding the fields are potentials is much easier if one starts in the frame of the charge!

Let $CS'$ be the inertial frame in which the charge is at rest. From § 3.2,[1]

$$\phi'(\boldsymbol{x}') = \frac{e}{r'},$$

and

$$A'^{\mu} = (\phi'(\boldsymbol{x}'), 0).$$

Let $CS$ be the inertial frame of the observer, and let $\boldsymbol{v} = (v, 0, 0)$. Then $CS$ and $CS'$ are related by a Lorentz boost; from Ch. 2 § 4.1,[2]

$$\begin{aligned}x' &= \gamma(x - vt)\\ y' &= y\\ z' &= z\end{aligned}$$

[1]The primes are not derivatives!

[2]Recall that $\gamma := \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

and, by boosting $A'^{\mu}$,

$$\begin{aligned}\underbrace{\phi}_{A^0} = \gamma \underbrace{\phi'}_{A'^0} = \gamma \frac{e}{r'} &= \gamma \frac{e}{\sqrt{x'^2 + y'^2 + z'^2}} \\ &= \gamma \frac{e}{\left(\gamma^2 (x - vt)^2 + y^2 + z^2\right)^{\frac{1}{2}}} \\ &= \frac{e}{\left((x - vt)^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2 + z^2)\right)^{\frac{1}{2}}}.\end{aligned}$$

This is the scalar potential due to the moving charge, which we can rewrite as

$$\boxed{\phi(\boldsymbol{x}, t) = \frac{e}{R^*}}, \quad \text{where } \boxed{R^* := \sqrt{(x - vt)^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2 + z^2)} = \frac{r'}{\gamma}}.$$

What about $\boldsymbol{A}$?

$$\boldsymbol{A}(\boldsymbol{x}, t) = \gamma \frac{\boldsymbol{v}}{c} \phi' = \frac{\boldsymbol{v}}{c} \phi(\boldsymbol{x}, t)$$

$$\Longrightarrow \boxed{\boldsymbol{A} = \frac{\boldsymbol{v}}{c} \frac{e}{R^*}}.$$

We calculate the fields using the same procedure. In $CS'$, we have

$$\boldsymbol{E}'(\boldsymbol{x}') = e \frac{\boldsymbol{x}'}{r'^3}, \quad \boldsymbol{B}'(\boldsymbol{x}') = 0.$$

We boost these, using the results from Ch. 2 § 4.2:

$$\begin{aligned}E_x &= E'_x = \frac{ex'}{(r')^3} = \frac{e}{\gamma^2} \frac{x - vt}{(R^*)^3} \\ E_y &= \gamma E'_y = \gamma \frac{ey'}{(r')^3} = \frac{e}{\gamma^2} \frac{y}{(R^*)^3} \\ E_z &= \frac{e}{\gamma^2} \frac{z}{(R^*)^3}.\end{aligned}$$

Thus,

$$\boxed{\boldsymbol{E} = \frac{e}{\gamma^2} \frac{\boldsymbol{R}}{(R^*)^3}}, \text{ where } \boxed{\boldsymbol{R}(\boldsymbol{x}, t) := (x - vt, y, z)}.$$

Note that $\boldsymbol{R}$ is the Galilean transformed $\boldsymbol{x}$.

What about $\boldsymbol{B}$? Again, from Ch. 2 § 4.2,

$$\begin{aligned}B_x &= B'_x = 0 \\ B_y &= -\gamma \frac{v}{c} E'_z = -\frac{v}{c} E_z \\ B_z &= \gamma \frac{v}{c} E'_y = \frac{v}{c} E_y\end{aligned}$$

$$\Longrightarrow \boxed{\boldsymbol{B}(\boldsymbol{x}, t) = \frac{\boldsymbol{v}}{c} \times \boldsymbol{E}(\boldsymbol{x}, t)}.$$

**Discussion of $\boldsymbol{E}(\boldsymbol{x}, t)$:**

Let $\theta$ be the angle between $\boldsymbol{v}$ and $\boldsymbol{R}$.[3]

$$\Longrightarrow \frac{\sqrt{y^2 + z^2}}{R} = \sin\theta \quad \Longrightarrow y^2 + z^2 = R^2 \sin^2\theta$$

[3]Reminder: in this section and elsewhere, a bold letter represents a vector, and the unbolded letter represents the magnitude of that vector: $\boldsymbol{R}$ vs. $R := |\boldsymbol{R}|$.

$$\Longrightarrow (R^*)^2 = R^2 - \frac{v^2}{c^2}\left(y^2+z^2\right) = R^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right)$$

$$\Longrightarrow \boxed{\boldsymbol{E}(\boldsymbol{x},t) = \frac{e}{\gamma^2}\frac{\boldsymbol{R}(\boldsymbol{x},t)}{R^3(\boldsymbol{x},t)}\frac{1}{\left[1-\frac{v^2}{c^2}\sin^2\theta(t)\right]^{\frac{3}{2}}}}.$$

Thus, for a fixed distance $R$ from the charge, $\boldsymbol{E}$ is minimized for $\theta = 0, \pi$; i.e., in the direction of the motion. The minimal value is

$$\underline{E_{\parallel} = \frac{e}{R^2}\left(1-\frac{v^2}{c^2}\right)}.$$

We can maximize $\boldsymbol{E}$ by taking $\theta = \pm\frac{\pi}{2}$; i.e., in the direction perpendicular to the motion. The maximal value is

$$\underline{E_{\perp} = \frac{e}{R^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}.$$

The field is no longer isotropic, but squeezed in the direction of the motion.. This is a manifestation of the Lorentz contraction.

*Remark* 1. We *could* have solved for the fields in this way: The 4-current in $CS'$ is

$$J'^{\mu} = \left(\rho'(x'), \boldsymbol{j}'(x')\right), \quad \text{with } \rho'(x') = e\delta(\boldsymbol{x}'), \boldsymbol{j}' = \boldsymbol{0}.$$

Thus, the observer in $CS$ sees

**charge density:** $\underline{\rho(\boldsymbol{x},t)} = \gamma\rho'(\boldsymbol{x}',t') = \gamma e\delta(\gamma(x-vt))\,\delta(y)\,\delta(z) = \underline{e\delta(\boldsymbol{R})}$.

**current density:** $\underline{\boldsymbol{j}(\boldsymbol{x},t)} = \gamma\frac{\boldsymbol{v}}{c}c\rho' = \boldsymbol{v}\rho = \underline{e\boldsymbol{v}\delta(\boldsymbol{R})}$.

Then we solve Maxwell's equations for this time-dependent 4-current. This is equivalent, but much harder to do!

## 3.5 Electrostatic interaction

Consider a time-independent charge density.

**Proposition 1.** *The energy of the electric field produced by* $\rho = \rho(\boldsymbol{x})$ *is*

$$\boxed{U = \frac{1}{2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \rho(\boldsymbol{x})\,\rho(\boldsymbol{y})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}.$$

*Proof.* From Ch. 2 § 3.6,

$$\begin{aligned} U &= \frac{1}{8\pi}\int_V d\boldsymbol{x}\,\boldsymbol{E}^2(\boldsymbol{x}) \\ &= -\frac{1}{8\pi}\int d\boldsymbol{x}\,\boldsymbol{E}\cdot\nabla\phi \\ &= -\frac{1}{8\pi}\underbrace{\int d\boldsymbol{x}\,\nabla(\boldsymbol{E}\phi)}_{\substack{\int d\boldsymbol{s}\cdot\boldsymbol{E}\phi \\ \to 0 \text{ as } V\to\infty}} + \frac{1}{8\pi}\int d\boldsymbol{x}\,\underbrace{(\nabla\cdot\boldsymbol{E})}_{4\pi\rho}\phi \\ &= \frac{1}{2}\int d\boldsymbol{x}\,\rho(\boldsymbol{x})\,\phi(\boldsymbol{x}) \\ &= \frac{1}{2}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\rho(\boldsymbol{x})\,\rho(\boldsymbol{y})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}, \end{aligned}$$

where the last line follows from Poisson's formula (§ 3.3) □

*Remark* 1. Let $\rho(\boldsymbol{x})$ be composed of $N$ localized charge distributions:$\rho(\boldsymbol{x}) = \sum_{\alpha=1}^{N} \rho^{(\alpha)}(\boldsymbol{x})$.

$$\begin{aligned} \Longrightarrow U &= \frac{1}{2}\sum_{\alpha,\beta}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \rho^{(\alpha)}(\boldsymbol{x}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho^{(\beta)}(\boldsymbol{y}) \\ &= \sum_{\alpha} U^{(\alpha)} + \sum_{\alpha\neq\beta} U^{(\alpha,\beta)}, \end{aligned}$$

where

$$\underline{U^{(\alpha)} := \frac{1}{2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \rho^{(\alpha)}(\boldsymbol{x}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho^{(\alpha)}(\boldsymbol{y})}$$

is called the *electrostatic self-energy* of charge distribution $\alpha$, and

$$\underline{U^{(\alpha,\beta)} := (1-\delta_{\alpha\beta})\frac{1}{2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \rho^{(\alpha)}(\boldsymbol{x}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho^{(\beta)}(\boldsymbol{y})}$$

is called the *electrostatic interaction energy* of localized charge distributions $\alpha$ and $\beta$ via the Coulomb interaction. As we shall see, the $\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}$ term in the self-energy is an issue.

*Remark* 2. Consider charged point particles: $\rho^{(\alpha)}(\boldsymbol{x}) = e_\alpha \delta\left(\boldsymbol{x} - \boldsymbol{x}^{(\alpha)}\right)$.

$$\Longrightarrow U^{(\alpha,\beta)} = (1-\delta_{\alpha\beta})\frac{1}{2}\frac{e_\alpha e_\beta}{\left|\boldsymbol{x}^{(\alpha)} - \boldsymbol{x}^{(\beta)}\right|}. \qquad \text{(Coulomb interaction)}$$

But $U^{(\alpha)}$ does not exist since we get $\frac{1}{0}$ once the $\delta$ functions are applied to the integrals.

*Remark* 3. Thus, the concept of a point charge leads to an infinite self-energy and makes no sense in classical electrodynamics. Only the interaction energy of point charges is physically meaningful.

*Remark* 4. One solution is to propose that maybe there aren't point charges; particles have some spacial extension. Let's estimate the smallest extension $r_0$ of a charge $e$ that still makes physical sense.

Let $\frac{e^2}{r_0} \cong mc^2$. Then $r_0 \cong \frac{e^2}{mc^2}$. For electrons, $r_0^e := \frac{e^2}{m_e c^2} \cong 2.8 \times 10^{-13}$ cm. This is called the *classical electron radius*. But experimental results place an upper limit on the radius of the electron to be $r_e < 10^{-20}$ cm.

We see that something is wrong with classical electrodynamics. Quantum mechanics is needed to resolve this issue. Ultimately, perfectly point-like things are not likely to be physical though. The Planck length may be the limit.

## 3.6 The law of Biot & Savart

**Proposition 1.** *A stationary*[a] *current density distribution* $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x})$ *leads to a vector potential*

$$\boxed{\boldsymbol{A} = \frac{1}{c}\int d\boldsymbol{y}\, \frac{\boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}}.$$

[a]That is, a "macroscopically stationary" or "steady" current.

*Proof.* From § 1.2, each component of $\boldsymbol{A}$ obeys Poisson's equation. The solution for each component is given by Poisson's formula. □

*Remark* 1. The proof in § 1.2 Proposition 1 required the use of the Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$)!

**Proposition 2. Law of Biot & Savart**
*The magnetic field generated by a static current density is*

$$\boxed{\boldsymbol{B}(\boldsymbol{x}) = -\frac{1}{c}\int d\boldsymbol{y}\, \frac{(\boldsymbol{x}-\boldsymbol{y}) \times \boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}}.$$

*Proof.* $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and

$$\begin{aligned}
\left(\nabla_{\boldsymbol{x}} \times \frac{\boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\right)_i &= \varepsilon_{ijk}\partial_j \frac{j_k(\boldsymbol{y})}{\left(\sum_{l=1}^{3}(x_l-y_l)^2\right)^{\frac{1}{2}}} \\
&= \varepsilon_{ijk} j_k(\boldsymbol{y}) \left(-\frac{1}{2}\right) \frac{2(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3} \\
&= -\varepsilon_{ijk}(\boldsymbol{x}-\boldsymbol{y})_j\, j_k(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3} \\
&= -\left[(\boldsymbol{x}-\boldsymbol{y}) \times \boldsymbol{j}(\boldsymbol{y})\right]_i \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}.
\end{aligned}$$

□

*Remark* 2. Notice the analogy between electrostatics and magnetostatics.

*Remark* 3. See 4.6 below for a discussion of the concept of a stationary current density

## 3.7 Magnetostatic interaction

Consider a time-independent current density.

**Proposition 1.** *The energy of the magnetic field produced by* $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x})$ *is*

$$\boxed{U = \frac{1}{2c^2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{x}) \cdot \boldsymbol{j}(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}.$$

**Lemma 1.** $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$

*Proof.*

$$\begin{aligned}
\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \partial_i \varepsilon_{ijk} A_j B_k \\
&= \varepsilon_{ijk}(\partial_i A_j) B_k + \varepsilon_{ijk} A_j (\partial_i B_k) \\
&= B_k \varepsilon_{kij} \partial_i A_j - A_j \varepsilon_{jik} \partial_i B_k \\
&= \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})
\end{aligned}$$

□

*Proof.* *(Of Proposition 1)*
From Ch. 2 § 3.6,

$$\begin{aligned}
U &= \frac{1}{8\pi}\int_V d\boldsymbol{x}\,\boldsymbol{B}^2(\boldsymbol{x}) \\
&= \frac{1}{8\pi}\int d\boldsymbol{x}\,\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) \\
&\overset{1.}{=} \frac{1}{8\pi}\underbrace{\int d\boldsymbol{x}\,\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})}_{\substack{\int d\boldsymbol{s}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \\ \to 0 \text{ as } V\to\infty}} + \frac{1}{8\pi}\int d\boldsymbol{x}\,\boldsymbol{A}\cdot\underbrace{(\nabla\times\boldsymbol{B})}_{\frac{4\pi}{c}\boldsymbol{j}} \\
&= \frac{1}{2c}\int d\boldsymbol{x}\,\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{j}(\boldsymbol{x}) \\
&\overset{2.}{=} \frac{1}{2c}\int d\boldsymbol{x}\,\boldsymbol{j}(\boldsymbol{x})\cdot\frac{1}{c}\int d\boldsymbol{y}\,\frac{\boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \frac{1}{2c^2}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\frac{\boldsymbol{j}(\boldsymbol{x})\cdot\boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}
\end{aligned}$$

1. From the lemma above.
2. From § 3.6 Proposition 1.

□

*Remark* 1. Let $\boldsymbol{j}(\boldsymbol{x})$ be composed of $N$ localized current distributions: $\boldsymbol{j}(\boldsymbol{x}) = \sum_{\alpha=1}^{N}\boldsymbol{j}^{(\alpha)}(\boldsymbol{x})$.

$$\begin{aligned}
\Longrightarrow U &= \frac{1}{2c^2}\sum_{\alpha,\beta}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\boldsymbol{j}^{(\alpha)}(\boldsymbol{x})\cdot\boldsymbol{j}^{(\beta)}(\boldsymbol{y})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \sum_{\alpha}U^{(\alpha)} + \sum_{\alpha\neq\beta}U^{(\alpha,\beta)},
\end{aligned}$$

where

$$\underline{U^{(\alpha)} := \frac{1}{2c^2}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\boldsymbol{j}^{(\alpha)}(\boldsymbol{x})\cdot\boldsymbol{j}^{(\alpha)}(\boldsymbol{y})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}$$

is called the *magnetostatic self-energy* of charge distribution $\alpha$, and

$$\underline{U^{(\alpha,\beta)} := (1-\delta_{\alpha\beta})\,\frac{1}{2}\int d\boldsymbol{x}\,d\boldsymbol{y}\,\boldsymbol{j}^{(\alpha)}(\boldsymbol{x})\cdot\boldsymbol{j}^{(\beta)}(\boldsymbol{y})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}}$$

is called the *magnetostatic interaction energy* of localized current distributions $\alpha$ and $\beta$ via the magnetostatic interaction.

# 4 Multipole expansion for static fields

## 4.1 The electric dipole moment

Consider a localized charge distribution $\rho = \rho(\boldsymbol{y})$.

**question:** What are the potential $\phi(\boldsymbol{x})$ and the field $\boldsymbol{E}(\boldsymbol{x})$ at a point $\boldsymbol{x}$ far from the charges?

Let $\rho(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| > r_0$; let $|\boldsymbol{x}| =: r \gg r_0$.

$$\Longrightarrow \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{1}{\sqrt{r^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + \boldsymbol{y}^2}} = \frac{1}{r}\Bigg(1 - 2\underbrace{\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r^2}}_{O\left(\frac{r_0}{r}\right)} + \underbrace{\frac{\boldsymbol{y}^2}{r^2}}_{O\left(\frac{r_0^2}{r^2}\right)}\Bigg)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left[1 + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r^2} + O\left(\frac{1}{r^2}\right)\right],$$

where the last step follows from the binomial approximation. Poisson's formula (§ 3.3) gives

$$\phi(\boldsymbol{x}) = \int d\boldsymbol{y}\,\frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} = \int d\boldsymbol{y}\,\rho(\boldsymbol{y})\,\frac{1}{r}\left[1 + \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r^2} + O\left(\frac{1}{r^2}\right)\right]$$

$$= \frac{1}{r}\int d\boldsymbol{y}\,\rho(\boldsymbol{y}) + \frac{\boldsymbol{x}}{r^3}\cdot\int d\boldsymbol{y}\,\boldsymbol{y}\rho(\boldsymbol{y}) + O\left(\frac{1}{r^3}\right)$$

**Proposition 1.** *For large distances $r$ from the localized charge distribution, the scalar potential has the form*

$$\boxed{\phi(\boldsymbol{x}) = \frac{Q}{r} + \frac{\boldsymbol{d}\cdot\boldsymbol{x}}{r^3} + O\left(\frac{1}{r^3}\right)},\ \textit{where}$$

$\underline{Q := \int d\boldsymbol{y}\,\rho(\boldsymbol{y})}$ *is the* total charge, *and*
$\underline{\boldsymbol{d} := \int d\boldsymbol{y}\,\boldsymbol{y}\rho(\boldsymbol{y})}$ *is the* electric dipole moment.

*Remark* 1. Analogous results hold for the gravitational potential of a localized mass distribution (PHYS 611).

*Remark* 2. If $Q = 0$, then $\boldsymbol{d}$ is independent of the origin of the coordinate system:
Let $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a}$ with $\boldsymbol{a} =$ const. Then in the new coordinate system we have $\rho'(\boldsymbol{y}) = \rho(\boldsymbol{y} - \boldsymbol{a})$

$$\Longrightarrow \underline{\boldsymbol{d}'} = \int d\boldsymbol{y}\,\boldsymbol{y}\rho'(\boldsymbol{y})$$

$$= \int d\boldsymbol{y}\,\boldsymbol{y}\rho(\boldsymbol{y}-\boldsymbol{a}) = \int d\boldsymbol{y}\,(\boldsymbol{y}+\boldsymbol{a})\,\rho(\boldsymbol{y}) = \underline{\boldsymbol{d} + Q\boldsymbol{a}}$$

$\therefore Q = 0 \Longrightarrow \boldsymbol{d}' = \boldsymbol{d}$.
If you ever get confused about this, consider a collection of point charges $e_\alpha$ at locations $\boldsymbol{x}_\alpha$:

$$\rho(\boldsymbol{y}) = \sum_\alpha e_\alpha \delta(\boldsymbol{y} - \boldsymbol{x}_\alpha)$$

Transform $CS \to CS'$ such that $\boldsymbol{x}'_\alpha = \boldsymbol{x}_\alpha + \boldsymbol{a}$. Then

$$\rho'(\boldsymbol{y}) = \sum_\alpha e_\alpha \delta(\boldsymbol{y} - a - \boldsymbol{x}_\alpha)$$

$$= \rho(\boldsymbol{y} - \boldsymbol{a}).$$

**Corollary 1.** *The field at large distances is*

$$\boxed{\boldsymbol{E}(\boldsymbol{x}) = Q\frac{\boldsymbol{x}}{r^3} + \frac{3(\hat{\boldsymbol{x}}\cdot\boldsymbol{d})\,\hat{\boldsymbol{x}} - \boldsymbol{d}}{r^3} + O\left(\frac{1}{r^4}\right)},$$

*where* $\hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$.

*Proof.* $\boldsymbol{E} = -\nabla\phi$ and $-\nabla\frac{Q}{r} = Q\frac{\boldsymbol{x}}{r^3}$ (see § 3.2),

$$\begin{aligned}\nabla\frac{\boldsymbol{d}\cdot\boldsymbol{x}}{r^3} &= \frac{1}{r^3}\nabla\left(\boldsymbol{d}\cdot\boldsymbol{x}\right)+\boldsymbol{d}\cdot\boldsymbol{x}\nabla\frac{1}{r^3}\\ &= \frac{\boldsymbol{d}}{r^3}+\boldsymbol{d}\cdot\boldsymbol{x}\left(-\frac{3}{2}\right)\left(\frac{1}{r^5}\right)(2\boldsymbol{x})\\ &= \frac{\boldsymbol{d}}{r^3}-\frac{3\left(\boldsymbol{d}\cdot\hat{\boldsymbol{x}}\right)\hat{\boldsymbol{x}}}{r^3}.\end{aligned}$$

□

*Remark* 3. For $Q = 0$, the leading contribution to the field falls off as $\frac{1}{r^3}$.

*Remark* 4. We can continue the expansion, with the next term being the quadrupole moment (a rank-2 tensor; see PHYS 611 and Problem #35). However, it is advantageous to introduce a more general concept.

## 4.2 Legendre functions and spherical harmonics

Note: the proofs in this section are omitted; see math books for proofs.

**Definition 1. *Legendre polynomials.*** The polynomials of degree $l$ defined by

$$\boxed{P_l(x) := \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l}, \text{ where } l = 0, 1, 2, \ldots$$

are called *Legendre polynomials.*

*Remark* 1. The first few Legendre polynomials are
$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}\left(3x^2-1\right)$

*Remark* 2. The $P_l(x)$ have the following properties $\forall l$:

(0) $P_l(1) = 1$

(i) $P_l(-x) = (-)^l P_l(x)$ — *parity*

(ii) $\left(1-x^2\right)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$ — *differential equation*

(iii) $P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$ — *recursion relation*

(iv) $\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \delta_{ll'}\frac{2}{2l+1}$ — *orthogonality*

*Remark* 3. $P_l(x)$ are members of a larger family of orthogonal polynomials known as the *classical orthogonal polynomials.*

**Theorem 1. Completeness of the Legendre polynomials**

*Any piecewise continuous and continuously differentiable function* $f : [-1,1] \to \mathbb{R}$ *can be expanded in Legendre polynomials as*

$$\boxed{f(x) = \sum_{l=0}^{\infty} f_l P_l(x)},$$

*where*

$$f_l = \left(\frac{2l+1}{2}\right)\int dx\, f(x)\, P_l(x)$$

*(from orthogonality).*

**Definition 2.** ***Associated Legendre functions.*** The functions (which are not polynomials now)

$$\boxed{P_l^m(x) := \frac{(-)^m}{2^l l!}\left(1-x^2\right)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^{l+m}\left(x^2-1\right)^l} \qquad \begin{array}{c} l = 0,1,2,\ldots \\ m = -l,-l+1,\ldots,l-1,l \end{array}$$

are called *associated Legendre functions.*

*Remark* 4. $P_l^0(x) = P_l(x)$ are the Legendre polynomials.

*Remark* 5. For fixed $l$, there are $2l+1$ functions $P_l^m$.

*Remark* 6. The first few $P_l^m(x)$ are
$P_0^0(x) = P_0(x) = 1$
$P_1^0(x) = P_1(x) = x$
$P_1^1(x) = -\sqrt{1-x^2}$
$P_1^{-1}(x) = \frac{1}{2}\sqrt{1-x^2}$

*Remark* 7. The $P_l^m$ have the properties:

(i) $P_l^m(\pm 1) = 0$ *zeroes*

(ii) $P_l^{-m}(x) = (-)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$ *m−symmetry*

(iii) $\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_l^m(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0$ *differential equation*

(iv) $\int_{-1}^{1} dx\, \underbrace{P_l^m(x)\, P_{l'}^m(x)}_{\text{same } m} = \delta_{ll'}\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$ *orthogonality*

**Definition 3.** ***Spherical harmonics.*** Consider a unit sphere. Let $\Omega = (\theta,\varphi)$ be a point on the sphere, and let $\eta = \cos\theta$ $(-1 \le \eta \le 1)$. The $\mathbb{C}-$valued functions defined on the sphere by

$$\boxed{Y_{lm}(\Omega) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi} P_l^m(\eta)}$$

are called *spherical harmonics.*

*Remark* 8. Different books define the normalization differently!

*Remark* 9. The first few spherical harmonics are
$Y_{00}(\Omega) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\Omega) = \sqrt{\frac{3}{4\pi}}\cos\theta$
$Y_{1,\pm 1}(\Omega) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\theta$

*Remark* 10. The $Y_{lm}$ have the properties:[4]

(i) $Y_{lm}^*(\Omega) = (-)^m Y_{l,-m}(\Omega)$ *complex conjugate*

(ii) $\left.\begin{aligned} &-i\frac{\partial}{\partial\varphi}Y_{lm}(\Omega) = mY_{lm}(\Omega) \\ &\Lambda Y_{lm}(\Omega) = -l(l+1)Y_{lm}(\Omega)\end{aligned}\right\}$ *differential equations*

(iii) $\int d\Omega\, Y_{lm}^*(\Omega)\, Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$ *orthogonality*

**Theorem 2. Completeness of spherical harmonics**

*Any piecewise-continuous and continuously differentiable function on the sphere, $f(\Omega)$, can be expanded in spherical harmonics:*

$$\boxed{f(\Omega) = \sum_{l,m} f_{lm} Y_{lm}(\Omega)},$$

*where the coefficients are given by*

$$\underline{f_{lm} = \int d\Omega\, f(\Omega)\, Y_{lm}^*(\Omega)}$$

*Remark* 11. This is often referred to by saying "the $Y_{lm}$ form a complete set on the sphere."

**Proposition 1. Addition theorem**

*Let $\Omega = (\theta, \varphi)$, $\Omega' = (\theta', \varphi')$, and let $\gamma$ be the angle between the two points:*

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

*Then*

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\Omega')\, Y_{lm}(\Omega)}.$$

**Corollary 1. Sum rule**

*For $\gamma = 0$, we have $\Omega = \Omega'$ and $P_l(1) = 1 = \frac{4\pi}{2l+1}\sum\limits_{m=-l}^{l} Y_{lm}^*(\Omega)\, Y_{lm}(\Omega)$.*

$$\Longrightarrow \boxed{\sum_{m=-l}^{l} |Y_{lm}(\Omega)|^2 = \frac{2l+1}{4\pi}}.$$

## 4.3 Separation of the Laplace operator in spherical coordinates

Consider the Laplace operator:

$$\nabla^2 =: \Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\Lambda,$$

with

$$\Lambda := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

[4] The symbol $\Lambda$ represents the angular part of the Laplacian $\nabla^2$ in spherical coordinates:
$\Lambda := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} = \frac{\partial}{\partial\eta}\left(1-\eta^2\right)\frac{\partial}{\partial\eta} + \frac{1}{1-\eta^2}\frac{\partial^2}{\partial\varphi^2}$,
where we have defined $\eta := \cos\theta$.

from § 4.2 Remark 10.

$$\Longrightarrow \nabla^2 f(r,\theta,\varphi) = \left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\Lambda\right) f(r,\theta\varphi) = \underbrace{\frac{1}{r}\partial_r^2 r}_{\substack{\text{acts on} \\ r \text{ only}}} f + \underbrace{\frac{1}{r^2}\Lambda}_{\substack{\text{acts on} \\ \theta,\varphi \\ \text{only}}} f$$

**Theorem 1.** *The differential equation for the function* $\psi(\boldsymbol{x})$

$$\boxed{\left[-\nabla^2 + V(r)\right]\psi(r,\theta,\varphi) = a(r,\theta,\varphi)} \tag{*}$$

*is solved by*

$$\boxed{\psi(r,\theta,\varphi) = \frac{1}{r}\sum_{l,m} u_{lm}(r)\, Y_{lm}(\Omega)},$$

*where* $u_{lm}(r)$ *is the solution of the ODE*

$$\boxed{\left(-\frac{d^2}{dr^2} + V_l(r)\right) u_{lm}(r) = r a_{lm}(r)} \tag{**}$$

*with*

$$\underline{V_l(r) := V(r) + \frac{l(l+1)}{r^2}}$$

*and*

$$\underline{a_{lm}(r) = \int d\Omega\, a(r,\theta,\varphi)\, Y^*_{lm}(\Omega)} \tag{†}$$

*Remark* 1. The Poisson Equation has the form $(*)$.

*Remark* 2. This theorem is also very useful in Quantum Mechanics.

*Proof. (Of Theorem 1)*
**ansatz:** $\psi(r,\theta,\varphi) = \frac{1}{r}\sum_{l,m} u_{lm}(r)\, Y_{lm}(\Omega)$

$$\begin{aligned}(*) \Longrightarrow a(r,\Omega) &= -\frac{1}{r}\partial_r^2 r \frac{1}{r}\sum_{l,m} u_{lm}(r)\, Y_{lm}(\Omega) - \frac{1}{r^2}\frac{1}{r}\sum_{l,m} u_{lm}(r)\, \underbrace{\Lambda Y_{lm}(\Omega)}_{-l(l+1)Y_{lm}} + V(r)\frac{1}{r}\sum_{l,m} u_{lm}(r)\, Y_{lm}(\Omega) \\ &= \sum_{l,m}\left[-\frac{1}{r}\partial_r^2 + \frac{l(l+1)}{r^3} + \frac{V(r)}{r}\right] u_{lm}(r)\, Y_{lm}(\Omega)\end{aligned}$$

(see § 4.2). By § 4.2 Theorem 2, any reasonably well behaved $a(r,\Omega)$ can be expanded in spherical harmonics:

$$a(r,\Omega) = \sum_{l,m} a_{lm}(r)\, Y_{lm}(\Omega),$$

with $a_{lm}(r)$ given by (†). Inserting this into the above equation:

$$\begin{aligned}\sum_{l,m} a_{lm}(r)\,Y_{lm}(\Omega) &= \sum_{l,m}\left[-\frac{1}{r}\partial_r^2+\frac{l(l+1)}{r^3}+\frac{V(r)}{r}\right]u_{lm}(r)\,Y_{lm}(\Omega)\,,\\ \Longrightarrow \underline{r a_{lm}(r)} &= \left[-\partial_r^2+\frac{l(l+1)}{r^2}+V(r)\right]u_{lm}(r)=\underline{\left(-\frac{d^2}{dr^2}+V_l(r)\right)u_{lm}(r)}\end{aligned}$$

which follows from the orthonormality of $Y_{lm}$. □

## 4.4 Expansion of harmonic functions in spherical harmonics

Consider harmonic functions, i.e., solutions of

$$\boxed{\nabla^2\phi(\boldsymbol{x})=0}, \qquad (*)$$

and assume that $\phi$ is twice continuously differentiable.

**Proposition 1.** *The most general solution of* $(*)$ *has the form*

$$\boxed{\phi(\boldsymbol{x})=\sum_{l,m}\left[\phi_{lm}^{+}(\boldsymbol{x})+\phi_{lm}^{-}(\boldsymbol{x})\right]},$$

*where*

$$\begin{aligned}\phi_{lm}^{+}(\boldsymbol{x}) &:= q_{lm}^{+}Y_{lm}(\Omega)\frac{1}{r^{l+1}}\\ \phi_{lm}^{-}(\boldsymbol{x}) &:= q_{lm}^{-}Y_{lm}(\Omega)\,r^{l}\end{aligned}$$

*with constant coefficients* $q_{lm}^{\pm}$.

*Proof.* Since $\nabla^2\phi=0$, we can expand $\phi$ using the theorem in § 4.3 with $V(r)=0$, $a(\boldsymbol{x})=0$.

$$\Longrightarrow \partial_r^2 u_{lm}(r)=\frac{l(l+1)}{r^2}u_{lm}(r)\,.$$

**ansatz:** $u_{lm}(r)=r^n$.

$$\Longrightarrow n(n-1)=l(l+1)$$

$$\Longrightarrow n=\begin{cases}l+1\\ -l\end{cases}$$

The two linearly independent solutions are therefore

$$u_{lm}(r)=\begin{cases}r^{-l}\\ r^{l+1}\end{cases}$$

$$\Longrightarrow \phi(\boldsymbol{x})=\sum_{l,m}\left(A\frac{1}{r^{l+1}}+Br^{l}\right)Y_{lm},$$

with $A$, $B$ arbitrary constants. □

*Remark* 1. $\phi_{lm}^{+}(\boldsymbol{x}\to\boldsymbol{0})\to\infty\quad\forall l,\qquad \phi_{lm}^{-}(\boldsymbol{x}\to\infty)\to\infty\quad\forall l>0.$

Thus, the only harmonic function that is finite at $r=0$ and $r\to\infty$ is the constant $l=0$ contribution (see § 1.2 Remark 5, § 3.1 Remark 2).

## 4.5 Multipole expansion of the electrostatic potential

**Lemma 1.**

$$\boxed{\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r_+}\sum_{l=0}^{\infty}\left(\frac{r_-}{r_+}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\Omega)\, Y_{lm}^*(\Omega')},$$

*where*

$$\boldsymbol{x} = (r, \Omega), \qquad r_+ = \max(r, r'),$$
$$\boldsymbol{x}' = (r', \Omega'), \qquad r_- = \min(r, r').$$

*Proof.* Let $\cos\gamma = \frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{rr'}$ (that is, $\gamma$ is the angle between $\boldsymbol{x}, \boldsymbol{x}'$).

$$\Longrightarrow |\boldsymbol{x}-\boldsymbol{x}'| = \sqrt{r^2 - 2rr'\cos\gamma + r'^2}.$$

*Case 1:* $\underline{r > r'}$

$$\begin{aligned}
\Longrightarrow \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} &\overset{1.}{=} \frac{1}{r}\left[1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}} \\
&= \frac{1}{r_+}\left[1 - 2\frac{r_-}{r_+}\cos\gamma + \left(\frac{r_-}{r_+}\right)^2\right]^{-\frac{1}{2}} \\
&\overset{2.}{=} \frac{1}{r_+}\sum_{l=0}^{\infty} f_l\left(\frac{r_-}{r_+}\right) P_l(\cos\gamma) \\
&\overset{3.}{=} \frac{1}{r_+}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1} f_l\left(\frac{r_-}{r_+}\right)\sum_{m=-l}^{l} Y_{lm}^*(\Omega')\, Y_{lm}(\Omega)
\end{aligned}$$

1. Note that it is only possible to factor $1/r$ if $r > r'$.
2. From Theorem 1 in § 4.2, we can expand the square root in terms of Legendre polynomials since $\cos\gamma \in [-1, 1]$.
3. From the *Addition Theorem* in § 4.2.

**Remaining question:** What is $f_l\left(\frac{r_-}{r_+}\right)$?

From § 4.4, $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$ is harmonic for $r > r'$ since

$$\nabla_{\boldsymbol{x}}^2 \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \nabla_{\boldsymbol{x}}^2 \frac{1}{r} = \frac{1}{r}\partial_r^2 r\frac{1}{r} = 0.$$

Furthermore, $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = O\left(\frac{1}{r}\right)$ for $r \to \infty$. By § 4.4, we know $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$ has the form $\phi_{lm}^+$ since it falls off as $\frac{1}{r}$. Thus,

$$\frac{1}{r} f_l\left(\frac{r'}{r}\right) = \frac{1}{r}\left(\frac{r'}{r}\right)^l c_l$$

for some constant $c_l$. For $\gamma = 0$,

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{r}\left[1 - 2\frac{r'}{r} + \left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}} = \frac{1}{r}\frac{1}{1-\frac{r'}{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \frac{1}{r}\sum_{l=0}^{\infty} f_l\left(\frac{r'}{r}\right)\underbrace{P_l(1)}_{1},$$

where the second to last step is the geometric series. Comparing the two equations above, $\implies c_l = 1$.

$$\implies \underline{f_l\left(\frac{r_-}{r_+}\right) = \left(\frac{r_-}{r_+}\right)^l}.$$

*Case 2:* $\underline{r' > r}$ analogous. □

**Proposition 1.** *The electrostatic potential of a localized charge distribution $\rho = \rho(\boldsymbol{x})$ (that is, $\rho(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > r_0$) can be written, for $|\boldsymbol{x}| > r_0$,*

$$\boxed{\phi(\boldsymbol{x}) = \sum_{l,m} \frac{Q_{lm}}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\Omega)},$$

*where*

$$\underline{Q_{lm} := \sqrt{\frac{4\pi}{2l+1}} \int_0^\infty dr\, r^2 r^l \int d\Omega\, \rho(r,\Omega)\, Y^*_{lm}(\Omega)}$$

*are the* multipole moments *of the charge distribution.*

*Proof.* From § 3.3 and inserting the lemma,[a]

$$\begin{aligned}
\phi(\boldsymbol{x}) &= \int d\boldsymbol{y}\, \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \int d\boldsymbol{y}\, \rho(\boldsymbol{y}) \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{y}{r}\right)^l \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\Omega_{\boldsymbol{x}})\, Y^*_{lm}(\Omega_{\boldsymbol{y}}) \\
&= \sum_{l,m} \frac{1}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\Omega_{\boldsymbol{x}}) \underbrace{\sqrt{\frac{4\pi}{2l+1}} \int dy\, y^2 y^l \int d\Omega\, \rho(y,\Omega_{\boldsymbol{y}})\, Y^*_{lm}(\Omega_{\boldsymbol{y}})}_{=:Q_{lm}}
\end{aligned}$$

□

[a] Note that $|\boldsymbol{x}| := r$, $|\boldsymbol{y}| := y$.

*Remark* 1.

For $l = 0$, the moment is

$$\underline{Q_{00}} = \sqrt{4\pi} \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega) \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y^*_{00}} = \underline{Q},$$

the total charge.

For $l = 1$, we have $Q_{1,-1}$, $Q_{10}$, $Q_{11}$:

$$Q_{1m} = \sqrt{\frac{4\pi}{3}} \int_0^\infty dr\, r^3 \int d\Omega\, \rho(r,\Omega) \left[\delta_{m0} \cos\theta - \delta_{m1} \frac{1}{\sqrt{2}} e^{-i\varphi} \sin\theta + \delta_{m,-1} \frac{1}{\sqrt{2}} e^{i\varphi} \sin\theta\right] \sqrt{\frac{3}{4\pi}}$$

$$\begin{aligned}
\Longrightarrow \underline{Q_{10}} &= \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega)\, r\cos\theta = \int d\boldsymbol{x} x_3 \rho(\boldsymbol{x}) = \underline{d_3} \\
\Longrightarrow \underline{Q_{11}} &= -\frac{1}{\sqrt{2}} \int_0^\infty dr\, r^2 \int d\Omega\, \rho(r,\Omega)\, e^{-i\varphi} r\sin\theta \\
&= -\frac{1}{\sqrt{2}} \int d\boldsymbol{x}\, \rho(\boldsymbol{x}) \left[r\sin\theta\cos\varphi - ir\sin\theta\sin\varphi\right] \\
&= -\frac{1}{\sqrt{2}} \int d\boldsymbol{x}\, \rho(\boldsymbol{x}) \left[x_1 - ix_2\right] = \underline{-\frac{1}{\sqrt{2}}(d_1 - id_2)} \\
\Longrightarrow \underline{Q_{1,-1}} &= \underline{\frac{1}{\sqrt{2}}(d_1 + id_2)}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \underline{d_1} &= \underline{\frac{1}{\sqrt{2}}(Q_{1,-1} - Q_{11}),} \\
\underline{d_2} &= \underline{\frac{1}{\sqrt{2}}(Q_{1,-1} + Q_{11}).}
\end{aligned}$$

## 4.6 Multipole expansion of the electrostatic interaction

Consider a charge density $\rho_<(\boldsymbol{x})$ confined to a region $R_<$ inside a sphere of radius $r_0$. Let $\rho_<(\boldsymbol{x})$ be subject to a charge density $\rho_>(\boldsymbol{y})$ confined to a region $R_>$ outside a sphere radius $R_0 > r_0$. What is the electrostatic interaction energy $U$ between these charge distributions?

From § 3.5,

$$\begin{aligned}
\underline{U} &= \frac{1}{2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \rho(\boldsymbol{x})\, \rho(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \frac{1}{2}\int_{R_<} d\boldsymbol{x}\, \rho_<(\boldsymbol{x}) \int_{R_>} d\boldsymbol{y}\, \rho_>(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{1}{2}\int_{R_>} d\boldsymbol{x}\, \rho_>(\boldsymbol{x}) \int_{R_<} d\boldsymbol{y}\, \rho_<(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \int_{R_<} d\boldsymbol{x}\, \rho_<(\boldsymbol{x}) \int_{R_>} d\boldsymbol{y}\, \rho_>(\boldsymbol{y}) \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \underline{\int_{R_<} d\boldsymbol{x}\, \rho_<(\boldsymbol{x})\, \phi_>(\boldsymbol{x}),}
\end{aligned}$$

where

$$\phi_>(\boldsymbol{x}) = \int_{R_>} d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \rho_>(\boldsymbol{y})$$

is the potential generated by the charges in $R_>$ at $\boldsymbol{x}$.[5]

If $R_0 \gg r_0$, $\phi_>(\boldsymbol{x})$ will vary slowly within $R_<$, so we can Taylor expand:

$$\phi_>(\boldsymbol{x}) = \phi_>(\boldsymbol{x}=\boldsymbol{0}) + \boldsymbol{x}\cdot\nabla\phi_>|_{\boldsymbol{x}=\boldsymbol{0}} + \frac{1}{2}x_i x_j \left.\frac{\partial^2}{\partial x_i \partial x_j}\phi_>\right|_{\boldsymbol{x}=\boldsymbol{0}} + \ldots$$

From § 1.1, $\phi_>(\boldsymbol{x})$ obeys Laplace's equation $\forall \boldsymbol{x} \in R_<$.

$$\Longrightarrow \delta_{ij} \left.\frac{\partial^2}{\partial x_i \partial x_j}\phi_>\right|_{\boldsymbol{x}=\boldsymbol{0}} = 0$$

$$\Longrightarrow \phi_>(\boldsymbol{x}) = \phi_>(\boldsymbol{x}=\boldsymbol{0}) + \boldsymbol{x}\cdot\nabla\phi_>|_{\boldsymbol{x}=\boldsymbol{0}} + \frac{1}{2}\left(x_i x_j - \frac{\boldsymbol{x}^2}{3}\delta_{ij}\right) \left.\frac{\partial^2}{\partial x_i \partial x_j}\phi_>\right|_{\boldsymbol{x}=\boldsymbol{0}} + \ldots$$

[5]Note that the choice to use $\phi_>$ as the source and $\rho_<$ as the test charge is arbitrary; we could have written $U$ in terms of $\rho_>$ and $\phi_<$.

**Definition 1.** Define the following:

$\phi_0 := \phi_> (\boldsymbol{x} = \boldsymbol{0})$ ...the potential $\phi_>$ at the origin

$\boldsymbol{E} := -\nabla\phi_> (\boldsymbol{x} = \boldsymbol{0})$ ...the field due to $\phi_>$ at the origin

$\phi_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}\phi_> (\boldsymbol{x} = \boldsymbol{0})$ ...the field gradients of $\phi_>$ at the origin

$$\implies \phi_> (\boldsymbol{x}) = \phi_0 - \boldsymbol{x} \cdot \boldsymbol{E} + \frac{1}{2}\left(x_i x_j - \frac{1}{3}\boldsymbol{x}^2 \delta_{ij}\right)\phi_{ij} + \dots$$

Now drop the subscripts, and denote $\rho := \rho_<$, $\phi := \phi_>$.

$$\begin{aligned} \implies U &= \int d\boldsymbol{x}\, \rho(\boldsymbol{x})\, \phi(\boldsymbol{x}) \\ &= \phi_0 \int d\boldsymbol{x}\, \rho(\boldsymbol{x}) - \boldsymbol{E} \cdot \int d\boldsymbol{x}\, \boldsymbol{x}\rho(\boldsymbol{x}) + \frac{1}{3}\phi_{ij}\frac{1}{2}\int d\boldsymbol{x}\, \rho(\boldsymbol{x})\left(3x_i x_j - \boldsymbol{x}^2\delta_{ij}\right) + \dots \end{aligned}$$

$$\implies \boxed{U = \phi_0 Q - \boldsymbol{E} \cdot \boldsymbol{d} + \frac{1}{3}\phi^{ij} Q_{ij} + \dots}$$

where

$\phi_0$, $\boldsymbol{E}$, $\phi^{ij}$ ...are the potential, electric field, and field gradient tensor due to $\rho >$ evaluated at the origin.

$Q$, $\boldsymbol{d}$, $Q_{ij}$ ...are the total charge, dipole moment, and quadrupole moments of $\rho_<$.

*Remark* 1. Alternatively, we can use the spherical harmonic expansion from § 4.4 to expand the potential:

$$\phi(\boldsymbol{x}) = \sum_{l,m} \phi^-_{lm}(\boldsymbol{x}) = \sum_{l,m} q^-_{lm} r^l Y_{lm}(\Omega),$$

where we have disregarded the $\phi^+_{lm}$ terms because they blow up at the origin.

$$\begin{aligned} \implies U &= \int d\boldsymbol{x}\, \rho(\boldsymbol{x})\, \phi(\boldsymbol{x}) \\ &= \int d\boldsymbol{x}\, \rho(\boldsymbol{x}) \sum_{l,m} q^-_{lm} r^l Y_{lm}(\Omega) \\ &= \sum_{l,m} q^-_{lm} \underbrace{\int dr\, r^2 r^l \int d\Omega\, \rho(r,\Omega)\, Y_{lm}(\Omega)}_{=\sqrt{\frac{2l+1}{4\pi}} Q^*_{lm}} \end{aligned}$$

$$\implies \boxed{U = \sum_{l,m} q^-_{lm} \sqrt{\frac{2l+1}{4\pi}} Q^*_{lm}},$$

where, as per § 4.5, $Q_{lm}$ are the multipole moments of the charge density $\rho(\boldsymbol{x}) := \rho_<(\boldsymbol{x})$ and the $q^-_{lm}$ are the coefficients of the expansion of the harmonic function $\phi(\boldsymbol{x}) := \phi_>(\boldsymbol{x})$ in spherical harmonics.

Which one is used depends on the symmetry of the problem.

## 4.7 The magnetic moment

From § 3.6, the Biot & Savart law gives the magnetic field resulting from a stationary current density. This requires an interpretation, as currents are produced by moving charges and hence are intrinsically time dependent.

**Definition 1.** ***Stationary current density.*** By *stationary current density* $\boldsymbol{j}(\boldsymbol{x})$, we mean the time average taken over a time $T$ large compared to all microscopic time-scales:

$$\boldsymbol{j}(\boldsymbol{x}) = \overline{\boldsymbol{j}(\boldsymbol{x},t)} := \frac{1}{T}\int_0^T dt\, \boldsymbol{j}(\boldsymbol{x},t)$$

**Example 1.** ***Current in a wire loop.***
$T$ must be much larger than the time it takes an electron to complete one revolution.

*Remark* 1. With this definition, M4 reduces to its static version upon time averaging, provided the electric field $\boldsymbol{E}$ as a function of time is bounded. That is,

$$\overline{\frac{\partial \boldsymbol{E}(\boldsymbol{x},t)}{\partial t}} = \frac{1}{T}\int_0^T dt\, \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{T}\left[\boldsymbol{E}(\boldsymbol{x},T) - \boldsymbol{E}(\boldsymbol{x},0)\right] \overset{T\to\infty}{\longrightarrow} \boldsymbol{0},$$

if $\boldsymbol{E}(\boldsymbol{x},t)$ is bounded.

$$(\text{M4}) \implies \underbrace{\overline{-\frac{1}{c}\partial_t \boldsymbol{E}}}_{0} + \overline{\nabla\times\boldsymbol{B}} = \boxed{\overline{\nabla\times\boldsymbol{B}} = \overline{\frac{4\pi}{c}\boldsymbol{j}}}.$$

Now consider the time averaged vector potential $\boldsymbol{A}(\boldsymbol{x}) := \overline{\boldsymbol{A}(\boldsymbol{x},t)}$ at large distances from a localized static current density given by

$$\boldsymbol{j}(\boldsymbol{y}) = \sum_\alpha \overline{e_\alpha \boldsymbol{v}_\alpha \delta(\boldsymbol{y}-\boldsymbol{x}_\alpha)}.$$

From § 3.6, the Biot & Savart law gives

$$\begin{aligned}
\underline{\boldsymbol{A}(\boldsymbol{x})} = \frac{1}{c}\int d\boldsymbol{y}\, \frac{\boldsymbol{j}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} &= \frac{1}{c}\sum_\alpha \overline{\frac{e_\alpha \boldsymbol{v}_\alpha}{|\boldsymbol{x}-\boldsymbol{x}_\alpha|}} \\
&\overset{1.}{=} \frac{1}{c}\sum_\alpha \overline{e_\alpha \boldsymbol{v}_\alpha \frac{1}{r}\left[1 + \frac{\boldsymbol{x}\cdot\boldsymbol{x}_\alpha}{r^2} + \ldots\right]} \\
&\overset{2.}{\approx} \underline{\frac{1}{c}\frac{1}{r^3}\sum_\alpha e_\alpha \overline{\boldsymbol{v}_\alpha(\boldsymbol{x}\cdot\boldsymbol{x}_\alpha)}}
\end{aligned}$$

1. Expanding $1/|\boldsymbol{x}-\boldsymbol{x}_\alpha|$ as per § 4.1.
2. The monopole contribution is zero by Remark 1:

$$\sum_\alpha \overline{e_\alpha \boldsymbol{v}_\alpha} = \overline{\frac{d}{dt}\sum_\alpha e_\alpha \boldsymbol{x}_\alpha} = 0$$

   since a static current density is assumed to be bounded. Here we also drop higher order terms.

We can rewrite the dipole term as follows:

$$\begin{aligned}
\sum_\alpha e_\alpha \boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x}) &= \sum_\alpha e_\alpha \dot{\boldsymbol{x}}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x}) \\
&= \frac{1}{2}\frac{d}{dt}\sum_\alpha e_\alpha \boldsymbol{x}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x}) + \frac{1}{2}\sum_\alpha e_\alpha\left[\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x}) - \boldsymbol{x}_\alpha(\boldsymbol{v}_\alpha\cdot\boldsymbol{x})\right],
\end{aligned}$$

by the product rule. Taking the time average,

$$\begin{aligned}
\implies \underline{\sum_\alpha e_\alpha \overline{\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})}} &= \frac{1}{2}\sum_\alpha e_\alpha \underbrace{\overline{\frac{d}{dt}\boldsymbol{x}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})}}_{0\text{ (bounded)}} + \frac{1}{2}\sum_\alpha e_\alpha\left[\overline{\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})} - \overline{\boldsymbol{x}_\alpha(\boldsymbol{v}_\alpha\cdot\boldsymbol{x})}\right] \\
&= \underline{\frac{1}{2}\sum_\alpha e_\alpha\left[\overline{\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})} - \overline{\boldsymbol{x}_\alpha(\boldsymbol{v}_\alpha\cdot\boldsymbol{x})}\right]}
\end{aligned}$$

$$\Longrightarrow \underline{\boldsymbol{A}(\boldsymbol{x})} = \underline{\frac{1}{2c}\frac{1}{r^3}\sum_\alpha e_\alpha\left[\overline{\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})} - \overline{\boldsymbol{x}_\alpha(\boldsymbol{v}_\alpha\cdot\boldsymbol{x})}\right]}.$$

**Definition 2.** ***Magnetic moment.*** The *magnetic moment* of the charges is defined as

$$\boxed{\boldsymbol{m} := \frac{1}{2c}\sum_\alpha e_\alpha\overline{(\boldsymbol{x}_\alpha\times\boldsymbol{v}_\alpha)}}.$$

**Proposition 1.** *The vector potential for large distances from the current density is given by the magnetic moment via*

$$\boxed{\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{r^3}\boldsymbol{m}\times\boldsymbol{x}} + O\left(\frac{1}{r^4}\right).$$

*Proof.*

$$\begin{aligned}\boldsymbol{m}\times\boldsymbol{x} &= \frac{1}{2c}\sum_\alpha e_\alpha\overline{(\boldsymbol{x}_\alpha\times\boldsymbol{v}_\alpha)}\times\boldsymbol{x}\\ &= \frac{1}{2c}\sum_\alpha e_\alpha\left[\overline{\boldsymbol{v}_\alpha(\boldsymbol{x}_\alpha\cdot\boldsymbol{x})} - \overline{\boldsymbol{x}(\boldsymbol{x}_\alpha\cdot\boldsymbol{v}_\alpha)}\right] = r^3\boldsymbol{A}.\end{aligned}$$

□

**Corollary 1.** *The magnetic field for large distances from the current density is*

$$\boxed{\boldsymbol{B}(\boldsymbol{x}) = \frac{3(\hat{\boldsymbol{x}}\cdot\boldsymbol{m})\hat{\boldsymbol{x}}-\boldsymbol{m}}{r^3}} + O\left(\frac{1}{r^4}\right).$$

*Proof.* Analogous to § 4.1. □

**Proposition 2.** *If all of the moving charges have the same charge-to-mass ratio $\frac{e_\alpha}{m_\alpha} =: \frac{e}{m}$, and if the motion is non-relativistic ($v_\alpha \ll c$), then the magnetic moment is proportional to the angular momentum of the system:*

$$\boxed{\boldsymbol{m} = \frac{e}{2mc}\boldsymbol{L}}. \qquad (*)$$

*Proof.* $\boldsymbol{L} := \sum_\alpha \boldsymbol{x}_\alpha\times\boldsymbol{p}_\alpha = \sum_\alpha m_\alpha\boldsymbol{x}_\alpha\times\boldsymbol{v}_\alpha$.

$$\Longrightarrow \underline{\boldsymbol{m}} := \frac{1}{2c}\sum_\alpha e_\alpha\overline{(\boldsymbol{x}_\alpha\times\boldsymbol{v}_\alpha)} = \frac{1}{2c}\sum_\alpha\frac{e_\alpha}{m_\alpha}m_\alpha\overline{(\boldsymbol{x}_\alpha\times\boldsymbol{v}_\alpha)} = \underline{\frac{e}{2mc}\boldsymbol{L}}.$$

□

*Remark* 2. The proportionality factor $\frac{e}{2mc}$ is called *gyromagnetic ratio*.

*Remark* 3. $(*)$ holds for orbital momentum $\boldsymbol{L}$ of particles, but *not* for the magnetic moment related to spin. For electrons,

$$\boldsymbol{m}_e = \frac{ge}{2mc}\boldsymbol{S}_e,$$

with $S_e = \frac{1}{2}\hbar$ the spin of the electron, $m$ the electron mass, and $g = 2.0023\cdots$ the so-called *g-factor*.

*Remark* 4. The g-factor was a mystery until the development of the Dirac equation, which predicts $g = 2$. The rest is accounted for by loop corrections in QED.

# Chapter 4

# Electromagnetic waves in vacuum

## 1 Plane electromagnetic waves

### 1.1 The wave equation

Consider vacuum: $J^{\mu}(x) = 0$ everywhere.

*Remark* 1. Any solutions to Maxwell's Equations must be time-dependent since in Ch. 3 § 1.1, § 1.2, we saw that in vacuum, static potentials obey Laplace's equation, which has only the trivial (zero) solution.

**Theorem 1. Wave equation.** *In vacuum (and with the Lorentz gauge), the 4-vector potential $A^{\mu}(x)$ obeys*

$$\boxed{\partial_{\nu}\partial^{\nu}A^{\mu}(x) = 0}. \qquad (*)$$

*Proof.* From Ch. 2 § 1.3,

$$\begin{aligned}
\partial_{\mu}F^{\mu\nu} &= \frac{4\pi}{c}J^{\nu} \overset{1.}{=} \underline{0} \\
&= \partial_{\mu}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right) \\
&\overset{2.}{=} \partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\underbrace{\partial_{\mu}A^{\mu}}_{0} \\
&= \underline{\partial_{\mu}\partial^{\mu}A^{\nu}}
\end{aligned}$$

1. We are considering vacuum.
2. In Lorentz gauge, $\partial_{\mu}A^{\mu} = 0$.

□

*Remark* 2. $(*)$ is called *wave equation.*

*Remark* 3. The operator

$$\boxed{\Box := \partial_{\nu}\partial^{\nu}}$$

is called *d'Alembert operator.* Explicitly,

$$\underline{\partial_{\nu}\partial^{\nu}} = g^{\mu\nu}\partial_{\nu}\partial_{\mu} = g^{\mu\nu}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} = \underline{\frac{1}{c^2}\partial_t^2 - \nabla^2}.$$

Some books define it as the negative of this.

*Remark* 4. The Lorentz gauge implies a Lorentz invariant relation between $\phi$ and $\boldsymbol{A}$:

$$\underline{0} = \partial_\mu A^\mu = \frac{\partial}{\partial x^\mu} A^\mu = \underline{\frac{1}{c}\partial_t \phi + \nabla \cdot \boldsymbol{A}}.$$

**Corollary 1.** *The electric and magnetic fields also obey the wave equation:*

$$\boxed{\Box \boldsymbol{E} = \Box \boldsymbol{B} = 0}. \tag{**}$$

*Proof.* From Ch. 2 § 3.4,

$$\begin{aligned}
\underline{\Box \boldsymbol{B}} &= \Box (\nabla \times \boldsymbol{A}) = \nabla \times (\Box \boldsymbol{A}) = \underline{\boldsymbol{0}} \\
\underline{\Box \boldsymbol{E}} &= \Box \left( -\nabla \phi - \frac{1}{c}\partial_t \boldsymbol{A} \right) = -\nabla (\Box \phi) - \frac{1}{c}\partial_t (\Box \boldsymbol{A}) = \underline{\boldsymbol{0}}
\end{aligned}$$

□

*Remark* 5. The Lorentz gauge still does not determine the potentials uniquely; in vacuum, one can always choose a gauge such that

$$\phi = 0 \implies \nabla \cdot \boldsymbol{A} = 0$$

(see Problem #39). However, this choice is *not* Lorentz invariant.

## 1.2 Plane waves

**Definition 1.** ***Plane waves.*** Solutions of the wave equation that depend on only one spacial coordinate are called *plane waves.*

Let $f(x,t)$ be any component of $\boldsymbol{E}$ or $\boldsymbol{B}$ or $A^\mu$. From § 1.1 $(*)$ or $(**)$,

$$\boxed{\left(\partial_t^2 - c^2 \partial_x^2\right) f(x,t) = 0}. \tag{*}$$

This is called the *plane wave equation* or *1D wave equation.*

**Theorem 1. d'Alembert solution.** *The most general solution of* $(*)$ *is*

$$\boxed{f(x,t) = f_1(x - ct) + f_2(x + ct)},$$

*where* $f_1$, $f_2$ *are arbitrary twice continuously differentiable functions of their arguments.*

*Proof.* We can write $(*)$ as

$$\underline{\left(\frac{1}{c}\partial_t - \partial_x\right)\left(\frac{1}{c}\partial_t + \partial_x\right) f(x,t) = 0.} \tag{†}$$

Define $\xi := x - ct$, $\eta := x + ct$. $\implies x = \frac{1}{2}(\xi + \eta)$, $t = -\frac{1}{2c}(\xi - \eta)$. Also define $\psi(\xi,\eta) := f(x,t)$.

$$\begin{aligned}
\implies \underline{\frac{1}{c}\partial_t f} &\overset{1.}{=} (\partial_\xi \psi)\frac{1}{c}\partial_t \xi + (\partial_\eta \psi)\frac{1}{c}\partial_t \eta \\
&= \underline{-\partial_\xi \psi + \partial_\eta \psi}, \\
\underline{\partial_x f} &= (\partial_\xi \psi)\partial_x \xi + (\partial_\eta \psi)\partial_x \eta \\
&= \underline{\partial_\xi \psi + \partial_\eta \psi}.
\end{aligned}$$

1. Inserting $f(x,t) =: \psi(\xi,\eta)$ and using the chain rule.

Inserting these relations into (†), we see

$$0 = -2\partial_\xi \underbrace{2\partial_\eta \psi\left(\xi,\eta\right)}_{=:a(\eta)}.$$

The bracketed term must not be a function of $\xi$ since, after a $\xi-$derivative, the result is 0. Integrate:

$$\Longrightarrow \quad \psi\left(\xi,\eta\right) = \int_{\eta_0}^{\eta} d\tilde{\eta}\, a\left(\tilde{\eta}\right) + b\left(\xi\right).$$

Note that both terms above are arbitrary functions. Let $f_1\left(\xi\right) := b\left(\xi\right),\ f_2\left(\eta\right) := \int_{\eta_0}^{\eta} d\tilde{\eta}\, a\left(\tilde{\eta}\right)$.

$$\Longrightarrow \quad \psi\left(\xi,\eta\right) = f_1\left(\xi\right) + f_2\left(\eta\right) = f_1\left(x-ct\right) + f_2\left(x+ct\right) = f\left(x,t\right).$$

□

*Remark* 1. PDEs in general have *whole classes* of functions as their solutions, in contrast to ODEs.

*Remark* 2. (Gotta insert this figure)
$f_1$ moves in the $+x$ direction with velocity $c$,
$f_2$ moves in the $-x$ direction with velocity $c$.
$f$ is a superposition of $f_1$, $f_2$.

## 1.3 Orientation of the fields

**Proposition 1.** *Consider a plane electromagnetic wave propagating in some direction* $\hat{\boldsymbol{n}}$*. Then* $\boldsymbol{E}$*,* $\boldsymbol{B}$*,* $\hat{\boldsymbol{n}}$ *are mutually perpendicular, and*

$$\boxed{\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{E}}.$$

*Proof.* By Problem #39, in vacuum we can always choose a gauge such that

$$\nabla \cdot \boldsymbol{A} = 0 \text{ and } \phi = 0.$$

Let $\hat{\boldsymbol{n}} = (1,0,0)$. $\Longrightarrow$ $\boldsymbol{A}\left(\boldsymbol{x},t\right) = \boldsymbol{A}\left(x,t\right)$, and have the wave travel in the $+x$ direction.

$$\Longrightarrow \quad \boldsymbol{A}\left(x,t\right) = \boldsymbol{A}\left(x-ct\right) = \boldsymbol{A}\left(t-\frac{x}{c}\right) = \boldsymbol{A}\left(u\right),$$

where $u := t - \frac{x}{c}$. Now, $\nabla \cdot \boldsymbol{A} = 0 \Longrightarrow \partial_x A_x = 0$. We also know

$$\begin{aligned} \Box \boldsymbol{A} = 0 \quad &\Longrightarrow \quad \partial_t^2 A_x = 0 \\ &\Longrightarrow \quad \partial_t A_x = \text{const.} \end{aligned}$$

If the constant were not zero, $E_x$ would not be zero since $\boldsymbol{E} = -\partial_t \boldsymbol{A}$. This would not fall off at infinity. Thus, assume $\partial_t A_x = 0 \Longrightarrow A_x = 0$. The wave solution does not fall off either, but its average vanishes.

$$\Longrightarrow \boldsymbol{A}\left(u\right) = \left(0, A_y\left(u\right), A_z\left(u\right)\right) \quad \cdots \quad \perp \hat{\boldsymbol{n}}$$

$$\Longrightarrow \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} = -\frac{1}{c}\partial_u \boldsymbol{A} \quad \cdots \quad \perp \hat{\boldsymbol{n}}$$

$$\begin{aligned} \implies \underline{\boldsymbol{B}} = \nabla \times \boldsymbol{A} &= (0, -\partial_x A_z, \partial_x A_y) \\ &= -\frac{1}{c}\partial_u (0, -A_z, A_y) \\ &= -\frac{1}{c}\partial_u (\hat{\boldsymbol{n}} \times \boldsymbol{A}) = -\frac{1}{c}\hat{\boldsymbol{n}} \times \partial_\mu \boldsymbol{A} = \underline{\hat{\boldsymbol{n}} \times \boldsymbol{E}}. \end{aligned}$$

□

**Corollary 1.** *The Poynting vector is given by*

$$\boxed{\boldsymbol{P}(\boldsymbol{x}, t) = cu(\boldsymbol{x}, t)\,\hat{\boldsymbol{n}}},$$

*where $u(\boldsymbol{x}, t)$ is the energy density of the fields.*

*Proof.* From Ch. 2 § 3.6,

$$\begin{aligned} \implies \underline{\boldsymbol{P}} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B} &= \frac{c}{4\pi}\boldsymbol{E} \times (\hat{\boldsymbol{n}} \times \boldsymbol{E}) \\ &\overset{1.}{=} \frac{c}{4\pi}\boldsymbol{E}^2 \hat{\boldsymbol{n}} \\ &\overset{2.}{=} \frac{c}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\hat{\boldsymbol{n}} \\ &= \underline{cu(\boldsymbol{x}, t)\,\hat{\boldsymbol{n}}}. \end{aligned}$$

1. Since $\boldsymbol{E} \perp \hat{\boldsymbol{n}}$.
2. Since $\boldsymbol{E}^2 = \boldsymbol{B}^2$.

□

*Remark* 1. The energy contained in the wave propagates with velocity $c$ in the direction $\hat{\boldsymbol{n}}$ perpendicular to the wave fronts.

## 1.4 Monochromatic plane waves

Consider the wave equation:

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) f(\boldsymbol{x}, t) = 0.$$

**Definition 1.** ***Monochromatic plane wave.*** A solution of the form

$$\boxed{f(\boldsymbol{x}, t) = f_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}}, \quad f_0 \in \mathbb{C},$$

is called a *monochromatic plane wave* with frequency $\omega$.

*Remark* 1. Problem #40 $\implies \omega^2 = c^2 \boldsymbol{k}^2 \iff f$ solves wave equation.

*Remark* 2. By the superposition principle from Ch. 2 § 5, if $f : \mathbb{R}^4 \to \mathbb{C}$ is a solution, then so are $\operatorname{Re} f$, $\operatorname{Im} f$.
What about $\boldsymbol{k}$?
*Case 1:* $\underline{k_x, k_y, k_z \in \mathbb{R}} \implies \omega \in \mathbb{R}$.
We can write $f_0 = |f_0|\, e^{-i\delta}$, where $\delta \in \mathbb{R}$. Then $\operatorname{Re} f$, $\operatorname{Im} f$ yield the two solutions:

$$\boxed{f(\boldsymbol{x}, t) = |f_0| \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t - \delta)},$$

$$\boxed{f(\boldsymbol{x}, t) = |f_0| \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t - \delta)}.$$

*Remark* 3. For fixed $\boldsymbol{x}$, $f$ is periodic in $t$ with period $T := \frac{2\pi}{\omega}$.

*Remark* 4. For fixed $t$, $f$ is periodic in space. Define

$$\underline{\varphi := \boldsymbol{k}\cdot\boldsymbol{x} - \omega t - \delta}$$

to be the *phase* of the wave.

$$f = \text{const.} \iff \varphi = \text{const.} \iff \underline{\boldsymbol{k}\cdot\boldsymbol{x} = \omega t + \delta}.$$

Thus, the surfaces of constant field are planes perpendicular to $\boldsymbol{k}$. $\boldsymbol{k}$ is called *wave vector*; $\lambda := \frac{2\pi}{|\boldsymbol{k}|}$ is called *wavelength*.

*Case 2:*
At least one of $k_i$ is not real, e.g., $k_x = \alpha + i\beta$.

$$\implies f(\boldsymbol{x}, t) = e^{i\alpha\lambda} e^{-\beta x} f_0 e^{i(k_y y + k_z z - \omega t)}$$

$$\implies f \to \infty \text{ if } x \to \mp\infty \text{ for } \beta \gtrless 0.$$

Thus, the solution is physically meaningful at most in a restricted space (e.g., total reflection at a surface).

*Case 3:* $\underline{\omega \notin \mathbb{R} \implies \boldsymbol{k} \notin \mathbb{R}^3}$.
This is the same as case 2, since $\omega^2 = c^2\boldsymbol{k}^2$.

## 1.5 Polarization of electromagnetic waves

Nothing we have derived prohibits $\boldsymbol{E}$, $\boldsymbol{B}$ from rotating about $\boldsymbol{k}$. We can express a monochromatic plane wave as

$$\begin{array}{c}\boxed{\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}} \\ \boxed{\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}}\end{array}, \text{ where } \omega^2 = c^2 k^2 \text{ with } k^2 := |\boldsymbol{k}|^2.$$

*Remark* 1. The direction of propagation $\hat{\boldsymbol{n}}$ from § 1.3 is

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} := \frac{\boldsymbol{k}}{|\boldsymbol{k}|} = \frac{\boldsymbol{k}}{\omega/c}.$$

From § 1.3 we also have

$$|\boldsymbol{E}_0| = |\boldsymbol{B}_0|$$

and $\boldsymbol{E}_0$, $\boldsymbol{B}_0$, $\boldsymbol{k}$ form a right-handed coordinate system.

Consider $\boldsymbol{E}$.[1] Let $\boldsymbol{E}_0^2 = |\boldsymbol{E}_0|^2 e^{-i2\alpha}$, and define $\boldsymbol{b} := \boldsymbol{E}_0 e^{i\alpha}$ with the property $\boldsymbol{b}^2 = \left|\boldsymbol{E}_0^2\right| \in \mathbb{R}$. Consider the physical solution

$$\boldsymbol{E}(\boldsymbol{x}, t) = \text{Re}\left[\boldsymbol{b} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t - \alpha)}\right],$$

where $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ with $\boldsymbol{b}_1 \perp \boldsymbol{b}_2$ since $\boldsymbol{b}^2 \in \mathbb{R}$. Let $\boldsymbol{k} = (k, 0, 0)$. $\implies \boldsymbol{b}_1 = (0, b_1, 0)$, $\boldsymbol{b}_2 = (0, 0, b_2)$,

$$\implies \begin{array}{rr} E_y = & b_1 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t - \alpha) \\ E_z = & -b_2 \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t - \alpha) \end{array} \implies \boxed{\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1}.$$

**Proposition 1.** *The* $\boldsymbol{E}-$*field vector moves on an ellipse; the same is true for the* $\boldsymbol{B}-$*field. This is called* elliptic polarization*.*

[1] This discussion is analogous for $\boldsymbol{B}-$field.

*Proof.* (above) □

*Remark* 2. Monochromatic plane waves are, in general, elliptically polarized.

*Remark* 3. Special cases:

$b_1 = b_2$ *circular polarization*

$b_1 = 0$ or $b_2 = 0$ *linear polarization*

*Remark* 4. Visualization:

## 1.6 The Doppler effect

Define the 4-wavevector $k^\mu := \left(\frac{\omega}{c}, \boldsymbol{k}\right) = (k_0, \boldsymbol{k})$. From problem #40, $k^\mu$ transforms as a Minkowski vector.

**Proposition 1.** *The 4-wavevector has zero length in Minkowski space:*

$$\boxed{k_\mu k^\mu = 0}.$$

*Proof.* $k_\mu k^\mu = \frac{\omega^2}{c^2} - \boldsymbol{k}^2 = 0$, from the wave equation. □

*Remark* 1. This implies our 4-wavevector lies on the *light cone* given by $k^0 = |\boldsymbol{k}|$. .

Consider an observer in a moving frame whose velocity forms some angle $\theta$ with propagation direction $\boldsymbol{k}$. What frequency does the moving observer see?

**Theorem 1. Doppler effect.** *If $\omega$ is the frequency of the wave observed in the rest frame, then the moving observer measures a different frequency $\omega'$ such that*

$$\boxed{\omega' = \gamma\omega\left(1 - \frac{v}{c}\cos\theta\right)}.$$

*Proof.* Lorentz boost 4-wavevector along $x$:

$$\implies \frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{v}{c}k_x\right).$$

But $k_x = |\boldsymbol{k}|\cos\theta = \frac{\omega}{c}\cos\theta$; insert this and factor $\frac{\omega}{c}$. □

*Remark* 2. The frequency shift given by $\left(1 - \frac{v}{c}\cos\theta\right)$ is called *linear Doppler effect.* The shift from $\gamma$ is called *quadratic Doppler effect.*

*Remark* 3. The quadratic Doppler effect is nonzero even for $\cos\theta = 0$; a manifestation of time dilation.

*Remark* 4. Consider a non-relativistic wave, e.g. a sound wave (density wave) in a fluid. The density fluctuation can be written

$$\delta n(\boldsymbol{x}, t) = a e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \text{ where } \omega = c_0 k$$

and $c_0$ is the phase velocity. Under a Galilean transformation,

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ t' &= t \end{aligned}$$

$$\begin{aligned} \implies \delta n(\boldsymbol{x}, t) &= a e^{i\left(k_x x' + k_x v t' + k_y y' - \omega t'\right)} \\ &= a e^{i\left(k_x x' + k_y y' - (\omega - k_x v) t'\right)} \end{aligned}$$

But $\omega - k_x v = \omega'$

$$\implies \omega' = \omega \left(1 - \frac{v}{c_0} \cos\theta\right).$$

Only the linear Doppler effect is observed, and there is *no* frequency shift for motion perpendicular to $\boldsymbol{k}$.

# 2 The wave equation as an initial value problem

## 2.1 The wave equation in Fourier space

From § 1.1, the general wave equation is

$$\boxed{\Box f(\boldsymbol{x}, t) = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) f(\boldsymbol{x}, t) = 0}. \qquad (*)$$

Take a spacial Fourier transform (Ch. 3 § 2), where

$$\underline{\hat{f}(\boldsymbol{k}, t) = \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x}, t)}$$

with back transform

$$\underline{f(\boldsymbol{x}, t) = \frac{1}{(2\pi)^3} \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{f}(\boldsymbol{k}, t).}$$

*Remark* 1. The generalized function concept implies this can be done for a large class of functions.

$$\begin{aligned} (*) \implies 0 &= \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) \frac{1}{(2\pi)^3} \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{f}(\boldsymbol{k}, t) \\ &= \frac{1}{(2\pi)^3} \int d\boldsymbol{k} \left(\frac{1}{c^2}\partial_t^2 - \boldsymbol{k}^2\right) \hat{f}(\boldsymbol{k}, t) \end{aligned}$$

But this integrand is positive definite.

$$\implies \boxed{\frac{d^2}{dt^2}\hat{f}(\boldsymbol{k}, t) + c^2\boldsymbol{k}^2 \hat{f}(\boldsymbol{k}, t) = 0}. \qquad (**)$$

An alternative way to see this is to multiply $(*)$ by $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$ and take the $\boldsymbol{x}$ integral:

$$\begin{aligned} \implies 0 &= \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) f(\boldsymbol{x}, t) \\ &= \frac{1}{c^2}\partial_t^2 \hat{f} + \boldsymbol{k}^2 \hat{f} \end{aligned}$$

(integrating by parts twice).

*Remark* 2. $(**)$ is an ODE for a harmonic oscillator with frequency[2]

$$\omega_{\boldsymbol{k}} = c\,|\boldsymbol{k}| =: ck.$$

*Remark* 3. The Fourier back transform theorem implies $(*)$ is equivalent to $(**)$.

[2]In this section, the notation $\omega_{\boldsymbol{k}}$ implies $\omega$ is a function of $\boldsymbol{k}$.

## 2.2 The general solution of the wave equation

The general solution of $(**)$ for $\hat{f}$ is

$$\boxed{\hat{f}(\boldsymbol{k},t) = a^0_{\boldsymbol{k}}\cos(\omega_{\boldsymbol{k}}t) + \frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\sin(\omega_{\boldsymbol{k}}t)},$$

where

$$\boxed{\begin{aligned} a^0_{\boldsymbol{k}} &:= \hat{f}(\boldsymbol{k},t=0) = \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x},t=0) \\ \dot{a}^0_{\boldsymbol{k}} &:= \partial_t \hat{f}(\boldsymbol{k},t)\Big|_{t=0} = \int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \partial_t f(\boldsymbol{x},t)|_{t=0} \end{aligned}}$$

**Theorem 1.** *The general solution of the wave equation is uniquely determined by the field and its time derivative at some initial time (WLOG $t=0$),*[a] *and is given by*

$$\boxed{f(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[a^0_{\boldsymbol{k}}\cos(\omega_{\boldsymbol{k}}t) + \frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\sin(\omega_{\boldsymbol{k}}t)\right]},$$

*with $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ and $a^0_{\boldsymbol{k}}$, $\dot{a}^0_{\boldsymbol{k}}$ defined above.*

[a] That is, $f(\boldsymbol{x},t=0)$ and $\partial_t f(\boldsymbol{x},t)|_{t=0}$.

**Corollary 1.** *The solution can also be written*

$$\boxed{f(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\left[f^+_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)} + f^-_{\boldsymbol{k}} e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_{\boldsymbol{k}}t)}\right]},$$

*where*

$$\boxed{f^{\pm}_{\boldsymbol{k}} := \frac{1}{2}\left(a^0_{\pm\boldsymbol{k}} \pm i\frac{1}{\omega_{\boldsymbol{k}}}\dot{a}^0_{\pm\boldsymbol{k}}\right)}.$$

*Proof.*

$$\begin{aligned} \underline{a^0_{\boldsymbol{k}}\cos(\omega_{\boldsymbol{k}}t) + \frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\sin(\omega_{\boldsymbol{k}}t)} &= a^0_{\boldsymbol{k}}\frac{1}{2}\left(e^{i\omega_{\boldsymbol{k}}t} + e^{-i\omega_{\boldsymbol{k}}t}\right) + \frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\frac{1}{2i}\left(e^{i\omega_{\boldsymbol{k}}t} - e^{-i\omega_{\boldsymbol{k}}t}\right) \\ &= \underbrace{\frac{1}{2}\left(a^0_{\boldsymbol{k}} + i\frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\right)}_{f^+_{\boldsymbol{k}}} e^{-i\omega_{\boldsymbol{k}}t} + \underbrace{\frac{1}{2}\left(a^0_{\boldsymbol{k}} - i\frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}}\right)}_{f^-_{-\boldsymbol{k}}} e^{i\omega_{\boldsymbol{k}}t} \end{aligned}$$

$$\begin{aligned} \Longrightarrow f(\boldsymbol{x},t) &= \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left[f^+_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}}t} + f^-_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}}t}\right] \\ &= \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\, f^+_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega_{\boldsymbol{k}}t} + \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\, f^-_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_{\boldsymbol{k}}t}, \end{aligned}$$

where in the last line, in the second term $\omega_{\boldsymbol{k}} = \omega_{-\boldsymbol{k}}$ is inserted. □

*Remark* 1. The general solution of the wave equation is a linear superposition of monochromatic plane waves with superposition amplitudes that are uniquely determined by the initial conditions $f(\boldsymbol{x},t=0)$ and $\dot{f}(\boldsymbol{x},t=0)$

# Chapter 5

# Electromagnetic radiation

**idea:** we have discussed

- static solutions of Maxwell's equations with sources (Ch. 3)
- dynamic solutions of Maxwell's equations in vacuum (Ch. 4).

Now we discuss

- dynamic solutions of Maxwell's equations with sources.

## 1 Review of potentials, gauges

### 1.1 Fields and potentials

Recall in Ch. 2 § 3.4, the fields $\boldsymbol{E}$, $\boldsymbol{B}$ (which are observable) can be obtained from potentials (that are not observable) via

$$\boxed{\begin{aligned}&\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t) - \frac{1}{c}\partial_t \boldsymbol{A}(\boldsymbol{x},t)\\ &\boldsymbol{B}(\boldsymbol{x},t) = \nabla\times\boldsymbol{A}(\boldsymbol{x},t)\end{aligned}}$$

*Remark* 1. The homogeneous Maxwell equations are automatically fulfilled by these.

*Remark* 2. From Ch. 2 § 3.1, $\phi$, $\boldsymbol{A}$ are the components of the 4-vector $A^\mu(x) = (\phi(x), \boldsymbol{A}(x))$.

**Proposition 1.** *The inhomogeneous Maxwell equations (M3, M4) are equivalent to four PDEs, which are the equations of motion (or field equations) for* $A^\mu(x)$*:*[a]

$$\boxed{\partial_\mu\partial^\mu A^\nu(x) - \partial^\nu\partial_\mu A^\mu(x) = \frac{4\pi}{c}J^\nu(x)} \qquad (*)$$

[a]Compare with Ch. 4 § 1.1.

*Proof.* From Ch. 2 § 1.3,

$$\begin{aligned}\frac{4\pi}{c}J^\nu &= \partial_\mu F^{\mu\nu}\\ &= \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu.\end{aligned}$$

□

**Corollary 1.** *In terms of $\phi$, $\boldsymbol{A}$, $(*)$ takes the form*

$$\boxed{\begin{aligned} \Box \boldsymbol{A} + \nabla\left(\frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}\right) &= \frac{4\pi}{c}\boldsymbol{j} \\ -\nabla^2\phi - \frac{1}{c}\partial_t\nabla\cdot\boldsymbol{A} &= 4\pi\rho \end{aligned}} \qquad (*')$$

*where* $\Box := \frac{1}{c^2}\partial_t^2 - \nabla^2$.

*Proof.* $J^\nu = (c\rho, \boldsymbol{j})$, $\partial^\mu := \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\partial_t, -\nabla\right)$, $\partial_\mu := \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \nabla\right)$.

$$\begin{aligned} \Longrightarrow \partial_\mu\partial^\mu &= \frac{1}{c^2}\partial_t^2 - \nabla^2 =: \Box, \\ \partial_\mu A^\mu &= \frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}. \end{aligned}$$

$\nu = 1, 2, 3$ in $(*)$ yields the first equation.

$\nu = 0$ in $(*)$ yields

$$\begin{aligned} \Box\phi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}\right) &= \frac{4\pi}{c}c\rho \\ = \cancel{\frac{1}{c^2}\partial_t^2\phi} - \nabla^2\phi - \cancel{\frac{1}{c^2}\partial_t^2\phi} - \frac{1}{c}\partial_t\nabla\cdot\boldsymbol{A} &= 4\pi\rho \end{aligned}$$

$\Box$

*Remark* 3. In the static case, $(*')$ simplifies to

$$\begin{aligned} \nabla^2\phi = 4\pi\rho \quad & \ldots \quad \text{Poisson's equation (Ch3 §1.1)} \quad \checkmark \\ \underbrace{-\nabla^2\boldsymbol{A} + \nabla(\nabla\cdot\boldsymbol{A})}_{=\nabla\times(\nabla\times\boldsymbol{A})=\nabla\times\boldsymbol{B}} = \frac{4\pi}{c}\boldsymbol{j} \quad & \ldots \quad \text{Fourth Maxwell equation} \quad \checkmark \end{aligned}$$

*Remark* 4. In vacuum and using Lorenz gauge, $(*)$ simplifies to

$$\Box A^\nu - \partial^\nu\underbrace{\partial_\mu A^\mu}_{=0} = 0 \quad \ldots \quad \text{wave equation (Ch4 §1.1)} \quad \checkmark$$

## 1.2 Gauge conventions

From Ch. 2 § 2.4, the potentials are not unique. We can choose certain constraints, called *gauge conventions*.

**Popular choices:**

(1) *Lorenz gauge* $\boxed{\partial_\mu A^\mu(x) = 0 = \frac{1}{c}\partial_t\phi + \nabla\cdot\boldsymbol{A}}$ Lorentz invariant

(2) *Coulomb gauge* $\boxed{\nabla\cdot\boldsymbol{A} = 0}$ (cf. Problem #19) not Lorentz invariant

*Remark* 1. Some books call (2) the *transverse gauge*, since $\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k}) = 0$ (from Fourier transforming), which implies $\boldsymbol{A} \perp \boldsymbol{k}$. Others call it *radiation gauge*.

*Remark* 2. Another possibility is to choose $\boxed{\phi(x) = 0}$. This is also sometimes called *radiation gauge*.

*Remark* 3. 4 potentials and 1 constraint (our choice of gauge) implies 3 potential fields uniquely determine the 6 fields $\boldsymbol{E}$, $\boldsymbol{B}$.

**Proposition 1.** *In Lorenz gauge, the field equations for the potentials § 1.1 (∗) becomes*

$$\boxed{\begin{aligned}\Box\boldsymbol{A} &= \frac{4\pi}{c}\boldsymbol{j}\\ \Box\phi &= 4\pi\rho\end{aligned}} \text{ or, } \boxed{\Box A^\mu = \frac{4\pi}{c}J^\mu}. \tag{$*$}$$

*Proof.* Lorenz gauge $\implies \partial_\mu A^\mu = 0$, $\therefore$ § 1.1 $(*) \implies (*)$. □

**Corollary 1.** *Once we choose Lorenz gauge, it is maintained under time evolution.*

*Proof.* $\Box\partial_\mu A^\mu = \partial_\mu\Box A^\mu = \frac{4\pi}{c}\underbrace{\partial_\mu J^\mu = 0}_{\text{continuity eq.}}$. □

*Remark* 4. From Ch. 2 § 2.1, $\partial_\mu J^\mu = 0$ is not an independent condition; it follows from the field equations.

**Proposition 2.** *In Coulomb gauge, the field equations become*

$$\boxed{\begin{aligned}\Box\boldsymbol{A} &= \frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c}\partial_t\nabla\phi\\ \nabla^2\phi &= -4\pi\rho\end{aligned}}. \tag{$**$}$$

*Proof.* § 1.1 $(*')$ with $\nabla\cdot\boldsymbol{A} = 0 \implies (**)$. □

**Corollary 2.** *Coulomb gauge is maintained under time evolution.*

*Proof.*

$$\begin{aligned}\underline{\Box\left(\nabla\cdot\boldsymbol{A}\right)} = \nabla\cdot(\Box\boldsymbol{A}) &\overset{1.}{=} \frac{4\pi}{c}\nabla\cdot\boldsymbol{j} - \frac{1}{c}\partial_t\nabla^2\phi\\ &\overset{2.}{=} \frac{4\pi}{c}\nabla\cdot\boldsymbol{j} + \frac{4\pi}{c}\partial_t\rho\\ &= \frac{4\pi}{c}\left(\nabla\cdot\boldsymbol{j} + \frac{1}{c}\partial_t c\rho\right)\\ &= \frac{4\pi}{c}\underbrace{\partial_\mu J^\mu = 0}_{\text{continuity eq.}}.\end{aligned}$$

1. Inserting $\Box\boldsymbol{A}$ from $(*')$.
2. Inserting $-\nabla^2\phi$ from $(*')$.

□

*Remark* 5. Which gauge to pick is a matter of choice. Different choices are convenient for different problems.

# 2 Green's functions; the Lorenz gauge

## 2.1 The concept of Green's functions

Consider an inhomogeneous wave equation:

$$\boxed{\Box f(\boldsymbol{x},t) = i(\boldsymbol{x},t)}, \tag{$*$}$$

with $i(\boldsymbol{x},t)$ a given inhomogeneity.

**Definition 1.** ***Green's function.*** A *Green's function* $G(\boldsymbol{x},t)$ for the PDE $(*)$ is a solution of

$$\boxed{\Box G(\boldsymbol{x},t) = \delta(\boldsymbol{x})\,\delta(t)}. \tag{$**$}$$

*Remark* 1. This is $(*)$ with a special inhomogeneity

$$i(\boldsymbol{x},t) = \delta(\boldsymbol{x})\,\delta(t) = \delta(x)\,\delta(y)\,\delta(z)\,\delta(t)\,.$$

**Proposition 1.** *Let $G(\boldsymbol{x},t)$ be a solution of $(**)$. Then*

$$\boxed{f(\boldsymbol{x},t) = \int d\boldsymbol{x}'\,dt'\,G(\boldsymbol{x}-\boldsymbol{x}',t-t')\,i(\boldsymbol{x}',t') =: (G\star i)(\boldsymbol{x},t)}$$

*is a solution of $(*)$.*

*Proof.*

$$\begin{aligned}
\Box f(\boldsymbol{x},t) &= \int d\boldsymbol{x}'\,dt'\,\Box G(\boldsymbol{x}-\boldsymbol{x}',t-t')\,i(\boldsymbol{x}',t')\\
&= \int d\boldsymbol{x}'\,dt'\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t')\,i(\boldsymbol{x}',t')\\
&= i(\boldsymbol{x},t)\,.
\end{aligned}$$

Note that this assumes we can interchange $\Box$, $\int$ which is allowed if $G$ is sufficiently well behaved. $\Box$

## 2.2 Green's functions for the wave equation

To find the form of Green's functions, take the Fourier transform of § 2.1 $(**)$ with respect to time. That is, take $\int dt\, e^{i\omega t}\,(**)$:[1]

$$\implies \delta(\boldsymbol{x})\underbrace{\int dt\, e^{i\omega t}\delta(t)}_{=1\text{ (Ch.3 §2.5)}} = \int dt\, e^{i\omega t}\frac{1}{c^2}\partial_t^2 G(\boldsymbol{x},t) - \nabla^2\underbrace{\int dt\, e^{i\omega t}G(\boldsymbol{x},t)}_{\text{define }=:G_\omega(\boldsymbol{x})}$$

$$\begin{aligned}
\implies \delta(\boldsymbol{x}) + \nabla^2 G_\omega(\boldsymbol{x}) &= \frac{1}{c^2}\int dt\, e^{i\omega t}\partial_t^2 G(\boldsymbol{x},t)\\
&\overset{1.}{=} \frac{(i\omega)^2}{c^2}\underbrace{\int dt\, e^{i\omega t}G(\boldsymbol{x},t)}_{=G_\omega(\boldsymbol{x})}
\end{aligned}$$

[1]Note that, by convention, we use $+i$ instead of $-i$ here.

1. Integrating by parts twice and assuming $G(\boldsymbol{x},t)$ falls off at $\pm\infty$, or, using Ch. 3 § 2.1 Proposition 3.

Thus, $G_\omega(\boldsymbol{x})$ obeys

$$\underline{-\left(\nabla^2+\frac{\omega^2}{c^2}\right)G_\omega(\boldsymbol{x})=\delta(\boldsymbol{x}).}$$

We solve this by taking the spacial Fourier transforms. Define

$$G_\omega(\boldsymbol{k}):=\int d\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}G_\omega(\boldsymbol{x}).$$

$$\Longrightarrow\left(\boldsymbol{k}^2-\frac{\omega^2}{c^2}\right)G_\omega(\boldsymbol{k})=1$$

$$\Longrightarrow \boxed{G_\omega(\boldsymbol{k})=\frac{1}{\boldsymbol{k}^2-\frac{\omega^2}{c^2}}}.$$

To find $G(\boldsymbol{x},t)$, we must back transform.

**spatial:**

$$\begin{aligned}\underline{G_\omega(\boldsymbol{x})} &= \int\frac{d\boldsymbol{k}}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}G_\omega(\boldsymbol{k})\\ &= \int\frac{d\boldsymbol{k}}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{1}{\boldsymbol{k}^2+\left(\frac{i\omega}{c}\right)^2}\\ &\overset{1.}{=} \underline{\frac{1}{4\pi}\frac{e^{\pm\frac{i\omega r}{c}}}{r}}\end{aligned}$$

1. From Problem #28, $\int\frac{d\boldsymbol{k}}{(2\pi)^3}\,e^{i\boldsymbol{k}\boldsymbol{x}}\frac{4\pi}{\boldsymbol{k}^2+\left(\frac{1}{r_0}\right)^2}=\frac{e^{-\frac{r}{r_0}}}{r}$, where $r_0=\pm\frac{c}{i\omega}$, $r=|\boldsymbol{x}|$.

**temporal:**

$$\begin{aligned}\underline{G(\boldsymbol{x},t)} &= \int\frac{d\omega}{2\pi}\,e^{-i\omega t}G_\omega(\boldsymbol{x})\\ &= \frac{1}{4\pi}\frac{1}{r}\int\frac{d\omega}{2\pi}\,e^{-i\omega t\pm\frac{i\omega r}{c}}\\ &= \frac{1}{4\pi r}\int\frac{d\omega}{2\pi}\,e^{-i\omega\left(t\mp\frac{r}{c}\right)}\\ &= \underline{\frac{1}{4\pi r}\delta\left(t\mp\frac{r}{c}\right).}\end{aligned}$$

**Theorem 1.** *The defining equation for the Green's functions (§ 2.1 (**)) has two solutions:*

$$\boxed{G_\pm(\boldsymbol{x},t)=\frac{1}{4\pi r}\delta\left(t\mp\frac{r}{c}\right)}\text{ where } r:=|\boldsymbol{x}|.$$

*Proof.* (above). □

*Remark* 1. Consider a point-like, time-dependent source

$$\underline{i(\boldsymbol{x},t)=\delta(\boldsymbol{x})\,i(t).}$$

From the proposition in § 2.1, the two solutions of the wave equation with this source are

$$\begin{aligned}\underline{f_\pm(\boldsymbol{x},t)} &= \int d\boldsymbol{x}'\,dt'\,\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\delta\left(t-t'\mp\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)\delta(\boldsymbol{x}')\,i(t')\\ &= \frac{1}{4\pi r}\int dt'\,\delta\left(t-t'\mp\frac{r}{c}\right)i(t')\\ &= \underline{\frac{1}{4\pi r}i\left(t\pm\frac{r}{c}\right)}.\end{aligned}$$

This implies that if the source $i(t')$ does something at a time $t'$, then the 4-potential response at position $\boldsymbol{x}$ occurs at a time

$$t = t' \pm \frac{r}{c}$$

for the solutions $f_\pm$.

**Definition 1.** Define:

$$\begin{aligned}&G_+ \text{ as "retarded Green's function"}\\ &G_- \text{ as "advanced Green's function"}\end{aligned}$$

**Axiom 4.** Causality; a physical response cannot precede the action of the source.

**consequence:** only the retarded solution is physical.

### 2.3 The retarded potentials

We can obtain the potentials by applying the proposition from § 2.1 and results from § 2.2 to the wave equations for $\boldsymbol{A}$, $\phi$:

$$\phi(\boldsymbol{x},t) = \int d\boldsymbol{x}'\,dt'\,\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\delta\left(t-t'-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)4\pi\rho(\boldsymbol{x}',t')$$

$$\Longrightarrow \boxed{\phi(\boldsymbol{x},t) = \int d\boldsymbol{y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho\left(\boldsymbol{y},t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}|\right)}. \qquad (*)$$

Analogously,

$$\Longrightarrow \boxed{\boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c}\int d\boldsymbol{y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{j}\left(\boldsymbol{y},t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}|\right)}. \qquad (**)$$

*Remark* 1. $(*)$, $(**)$ are called *retarded potentials.*

*Remark* 2. The time delay $\Delta t = \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}$ corresponds to the time it takes the wave to travel from point $\boldsymbol{y}$ to $\boldsymbol{x}$ with velocity $c$.

*Remark* 3. $(*)$, $(**)$ are analogous to Poisson's formula in the static case (cf. Ch. 3 § 3.6). The new concept from time dependence is retardation from finite speed of propagation.

## 3 Radiation by time-dependent sources

### 3.1 Asymptotic potentials and fields

Consider retarded potentials (§ 2.3 $(*)$, $(**)$) at large distances $r = |\boldsymbol{x}|$ from the sources.

We can expand $|\boldsymbol{x}-\boldsymbol{y}|$:

$$\begin{aligned}|\boldsymbol{x}-\boldsymbol{y}| &= \sqrt{r^2-2\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}^2}\\ &= r\sqrt{1-2\frac{\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}{r}+O\left(\frac{1}{r^2}\right)}\\ &= r-\hat{\boldsymbol{x}}\cdot\boldsymbol{y}+O\left(\frac{1}{r}\right),\end{aligned}$$

where $\hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{r}$.

$$\implies \boxed{\phi(\boldsymbol{x},t)=\frac{1}{r}\int d\boldsymbol{y}\,\rho(\boldsymbol{y},t_r)}+O\left(\frac{1}{r^2}\right),$$

where $\underline{t_r := t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}| \approx t-\frac{r}{c}+\frac{1}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}$. Analogously,

$$\implies \boxed{\boldsymbol{A}(\boldsymbol{x},t)=\frac{1}{cr}\int d\boldsymbol{y}\,\boldsymbol{j}(\boldsymbol{y},t_r)}+O\left(\frac{1}{r^2}\right).$$

*Remark* 1. We keep only leading terms for $r\to\infty$, which are of $O\left(\frac{1}{r}\right)$.

*Remark* 2. How many terms to keep in the time argument $t_r$ of $\rho$, $\boldsymbol{j}$ depends on how rapidly the sources are changing. If $L$ is the linear dimension of the source, and the source changes appreciably on a time scale $\Delta t=\frac{L}{c}$, then the term $\frac{1}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}$ may be important.

Before deriving the asymptotic forms of $\boldsymbol{E}$, $\boldsymbol{B}$, we prove two useful lemmas.

**Lemma 1.**

$$\boxed{\nabla\frac{1}{r}f(t_r)=-\frac{1}{c}\hat{\boldsymbol{x}}\frac{1}{r}\partial_t f(t_r)}+O\left(\frac{1}{r^2}\right)$$

*Note: In this section,* $\partial_t f(t_r) := (\partial_t f)|_{t_r}$ *and* $\nabla\frac{1}{r}f(t_r) := \nabla\left(\frac{1}{r}f(t_r)\right)$.

*Proof.*

$$\begin{aligned}\nabla\frac{1}{r}f(t_r) &\overset{1.}{=} \underbrace{\left(\nabla\frac{1}{r}\right)f(t_r)}_{O(1/r^2)}+\frac{1}{r}\nabla(f(t_r))\\ &\overset{2.}{=} \frac{1}{r}(\partial_t f)(t_r)\nabla t_r+O\left(\frac{1}{r^2}\right)\\ &\overset{3.}{=} \frac{1}{r}(\partial_t f)(t_r)\left(-\frac{1}{c}\right)\nabla\sqrt{x^2+y^2+z^2}+O\left(\frac{1}{r^2}\right)\\ &= -\frac{1}{cr}\underbrace{\frac{\boldsymbol{x}}{r}}_{\hat{\boldsymbol{x}}}\partial_t f(t_r)+O\left(\frac{1}{r^2}\right)\end{aligned}$$

1. Product rule.
2. Chain rule.
3. $\nabla t_r=-\frac{1}{c}\nabla\sqrt{x^2+y^2+z^2}$

□

**Lemma 2.**

$$\boxed{\partial_t \rho(\boldsymbol{y}, t_r) = -\nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t_r) + \frac{1}{c}\hat{\boldsymbol{x}} \cdot \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)} + O\left(\frac{1}{r}\right)$$

*Proof.* By the continuity equation (Ch. 2 § 2.1),

$$\Longrightarrow \partial_t \rho(\boldsymbol{x}, t) = -\nabla_{\boldsymbol{x}} \cdot \boldsymbol{j}(\boldsymbol{x}, t)$$

$$\Longrightarrow (\partial_t \rho(\boldsymbol{y}, t))_{t=t_r} = -(\nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t))_{t=t_r}$$

But

$$\begin{aligned} \nabla_{\boldsymbol{y}} \cdot (\boldsymbol{j}(\boldsymbol{y}, t_r)) &\overset{1.}{=} (\nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t))_{t=t_r} + (\partial_t \boldsymbol{j})(\boldsymbol{y}, t_r) \cdot \nabla_{\boldsymbol{y}} t_r \\ &\overset{2.}{=} (\nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t))_{t=t_r} + \frac{1}{c}\hat{\boldsymbol{x}} \cdot (\partial_t \boldsymbol{j})(\boldsymbol{y}, t_r) + O\left(\frac{1}{r}\right) \end{aligned}$$

$$\begin{aligned} \Longrightarrow (\partial_t \rho(\boldsymbol{y}, t))_{t=t_r} =: \partial_t \rho(\boldsymbol{y}, t_r) &= -\nabla_{\boldsymbol{y}} \cdot (\boldsymbol{j}(\boldsymbol{y}, t_r)) + \frac{1}{c}\hat{\boldsymbol{x}} \cdot (\partial_t \boldsymbol{j})(\boldsymbol{y}, t_r) + O\left(\frac{1}{r}\right) \\ &=: -\nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t_r) + \frac{1}{c}\hat{\boldsymbol{x}} \cdot \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r) + O\left(\frac{1}{r}\right). \end{aligned}$$

1. Chain rule.
2. Recall $t_r := t - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|$

$$\begin{aligned} \Longrightarrow \nabla_{\boldsymbol{y}} t_r &= -\frac{1}{c}\nabla_{\boldsymbol{y}} |\boldsymbol{y} - \boldsymbol{x}| \\ &= -\frac{1}{c}\frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|} = \frac{1}{c}\frac{\boldsymbol{x}}{r\sqrt{1 - 2\hat{\boldsymbol{x}} \cdot \boldsymbol{y} + \frac{\boldsymbol{y}^2}{r^2}}} - \frac{\boldsymbol{y}}{r\sqrt{\cdots}} = \frac{1}{c}\hat{\boldsymbol{x}}\left(1 + O\left(\frac{1}{r}\right)\right) + O\left(\frac{1}{r}\right). \end{aligned}$$

□

**Proposition 1.** *Far from the sources, the fields are*

$$\boxed{\begin{aligned} &\boldsymbol{B}(\boldsymbol{x}, t) = -\frac{1}{c^2}\frac{\hat{\boldsymbol{x}}}{r} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r) \\ &\boldsymbol{E}(\boldsymbol{x}, t) = -\hat{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x}, t) \end{aligned}} \cdots + O\left(\frac{1}{r^2}\right).$$

*Remark* 3. This implies $\boldsymbol{E}^2 = \boldsymbol{B}^2$, and $\hat{\boldsymbol{x}} \perp \boldsymbol{E} \perp \boldsymbol{B}$, forming a right-handed orthogonal set.

*Remark* 4. The fields fall off as $\frac{1}{r}$ as opposed to $\frac{1}{r^2}$ in static solutions.

*Proof. (of proposition).*

From § 1.1, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, so by the equation for asymptotic $\boldsymbol{A}$,

$$\begin{aligned}
\Longrightarrow B_i &= \varepsilon_{ijk}\partial_j \frac{1}{r}\frac{1}{c}\int d\boldsymbol{y}\, j_k(\boldsymbol{y}, t_r) \\
&\overset{1.}{=} \varepsilon_{ijk}\left(-\frac{1}{c^2}\right)\hat{x}_j \frac{1}{r}\int d\boldsymbol{y}\, \left(\partial_t j_k(\boldsymbol{y}, t)\right)_{t=t_r} \\
\Longrightarrow \boldsymbol{B}(\boldsymbol{x}, t) &= -\frac{1}{c^2}\frac{\hat{\boldsymbol{x}}}{r} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)
\end{aligned}$$

1. By lemma 1, $\partial_j \frac{1}{r} f(t_r) = -\frac{1}{c}\hat{x}_j \frac{1}{r}\partial_t f(t_r)$.

From § 1.1, $\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\partial_t \boldsymbol{A}$, so by the equations for asymptotic $\phi$, $\boldsymbol{A}$,

$$\begin{aligned}
\Longrightarrow \boldsymbol{E} &= -\nabla \frac{1}{r}\int d\boldsymbol{y}\, \rho(\boldsymbol{y}, t_r) - \frac{1}{c}\partial_t \frac{1}{cr}\int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t_r) \\
&\overset{1.}{=} \frac{1}{c}\frac{\hat{\boldsymbol{x}}}{r}\int d\boldsymbol{y}\, \partial_t \rho(\boldsymbol{y}, t_r) - \frac{1}{c^2 r}\int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r) \\
&\overset{2.}{=} -\frac{1}{c}\frac{\hat{\boldsymbol{x}}}{r}\underbrace{\int d\boldsymbol{y}\, \nabla_{\boldsymbol{y}} \cdot \boldsymbol{j}(\boldsymbol{y}, t_r)}_{=\int_{\mathbb{R}^3} d\boldsymbol{S}\cdot\boldsymbol{j} \to 0} + \frac{1}{c}\frac{\hat{\boldsymbol{x}}}{r}\int d\boldsymbol{y}\, \frac{1}{c}\hat{\boldsymbol{x}} \cdot \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r) - \frac{1}{c^2 r}\int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r) \\
&= \frac{1}{c^2 r}\int d\boldsymbol{y}\, \left[\hat{\boldsymbol{x}}\left(\hat{\boldsymbol{x}} \cdot \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)\right) - \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)\right] \\
&\overset{3.}{=} \frac{1}{c^2 r}\int d\boldsymbol{y}\, \hat{\boldsymbol{x}} \times \left(\hat{\boldsymbol{x}} \times \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)\right) \\
&= -\hat{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x}, t).
\end{aligned}$$

1. By lemma 1, and since $\partial_t = \partial_{t_r}$.
2. By lemma 2.
3. Using the vector identity:

$$\begin{aligned}
(\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b}))_i &= \varepsilon_{ijk} a_j \varepsilon_{klm} a_l b_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j a_l b_m \\
&= a_i (\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{a}^2 b_i
\end{aligned}$$

□

*Remark* 5. A time-dependent localized current density leads to time-dependent fields everywhere in space (with proper retardation to account for signal travel time). This phenomenon is called *radiation.*

*Remark* 6. Far from the source, the radiation fields $\boldsymbol{E}$, $\boldsymbol{B}$ ...
(i) falls off as $\frac{1}{r}$
(ii) are perpendicular to one another and to the radius vector from source to observer (because we are far enough away that the waves are approximately plane waves).

*Remark* 7. The source must provide the field energy; there is steady power loss at the source.

### 3.2 The radiated power

From Ch. 2 § 3.6, the energy-current density of the fields is given by the Poynting vector:

$$\underline{\boldsymbol{P}(\boldsymbol{x}, t) = \frac{c}{4\pi}\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{B}(\boldsymbol{x}, t).}$$

*Remark* 1. $\boldsymbol{E} \perp \boldsymbol{B} \perp \hat{\boldsymbol{x}} \implies \boldsymbol{P} \parallel \hat{\boldsymbol{x}}$.

*Remark* 2. $[\boldsymbol{P}] = \text{energy per time and area} = \mathrm{erg}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} = \mathrm{g}\,\mathrm{cm}\,\mathrm{s}^{-3}$.

*Remark* 3. $\hat{\boldsymbol{x}} \cdot \boldsymbol{P} =$ power per unit area. Denote by $\mathscr{P}$ the total radiated power. Then the power radiated per solid angle is given by

$$\begin{aligned} \frac{d\mathscr{P}}{d\Omega} &= \hat{\boldsymbol{x}} \cdot \boldsymbol{P} dA \\ &= \left(\hat{\boldsymbol{x}} \cdot \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{B}\right) \left(r^2 d\Omega\right) \\ &= \frac{c}{4\pi} r^2 \hat{\boldsymbol{x}} \cdot (\boldsymbol{B} \times (\hat{\boldsymbol{x}} \times \boldsymbol{B})) \end{aligned}$$

But

$$\begin{aligned} \hat{\boldsymbol{x}} \cdot (\boldsymbol{B} \times (\hat{\boldsymbol{x}} \times \boldsymbol{B})) &= \hat{x}_i \varepsilon_{ijk} B_j \varepsilon_{klm} \hat{x}_l B_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \hat{x}_i B_j \hat{x}_l B_m \\ &= \boldsymbol{B}^2 - \underbrace{\cancel{(\boldsymbol{B} \cdot \hat{\boldsymbol{x}})^2}}_{\boldsymbol{B} \perp \hat{\boldsymbol{x}}} \\ &= \boldsymbol{B}^2 \end{aligned}$$

$$\begin{aligned} \implies \frac{d\mathscr{P}}{d\Omega} &= \frac{c}{4\pi} r^2 \boldsymbol{B}^2 \\ &= \frac{c}{4\pi} \cancel{r^2} \left(\frac{1}{c^2 \cancel{r}}\right)^2 \left(\hat{\boldsymbol{x}} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}\left(\boldsymbol{y}, t_r\right)\right)^2 \\ &= \frac{1}{4\pi c^3} \left(\hat{\boldsymbol{x}} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}\left(\boldsymbol{y}, t_r\right)\right)^2 \end{aligned}$$

**Theorem 1.** *The power radiated by the source per solid angle is*

$$\boxed{\frac{d\mathscr{P}}{d\Omega} = \frac{1}{4\pi c^3} \left(\hat{\boldsymbol{x}} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}\left(\boldsymbol{y}, t_r\right)\right)^2}.$$

*Proof.* *(above).* □

*Remark* 4. Power $\propto (\text{fields})^2$, and fields $\propto \frac{1}{r}$: there is nonzero power per solid angle even as $r \to \infty$.

**Corollary 1.** *The total power radiated is*

$$\boxed{\mathscr{P} = \int d\Omega \frac{d\mathscr{P}}{d\Omega}}.$$

## 3.3 Radiation by an accelerated charged point particle

Consider a point particle with charge $e$ moving with non-relativistic velocity $v \ll c$, on a trajectory $\boldsymbol{R}(t)$.

**current density:** $\underline{\boldsymbol{j}(\boldsymbol{y}, t) = e\boldsymbol{v}(t)\,\delta(\boldsymbol{y} - \boldsymbol{R}(t))}$ where $\boldsymbol{v}(t) := \dot{\boldsymbol{R}}(t)$

$\boldsymbol{y} = \boldsymbol{R}(t)$ is time dependent and needs to be taken at the **retarded time** (see §2.3):

$$\begin{aligned}\underline{t_r} &= t - \frac{1}{c}\left|\boldsymbol{x} - \boldsymbol{y}\right| \\ &\overset{1.}{=} t - \frac{1}{c}\left|\boldsymbol{x} - \boldsymbol{R}(t_r)\right| \\ &\overset{2.}{\approx} t - \frac{r}{c} + \frac{1}{c}\hat{\boldsymbol{x}} \cdot \boldsymbol{R}(t_r) \\ &\overset{3.}{\approx} \underline{t - \frac{r}{c} =: t_e}\end{aligned}$$

1. The second line is an exact implicit equation for the retarded time $t_r$.
2. The third line is valid at large distances $r \gg |\boldsymbol{y}|$ from the source.
3. The third line is valid for nonrelativistic motion: $\frac{d}{dt}(t - r/c) = 1$, and $\frac{d}{dt}\boldsymbol{R} = \boldsymbol{v} \Rightarrow \frac{1}{c}\hat{\boldsymbol{x}} \cdot \boldsymbol{R}(t')$ is small of order $v/c$ compared to $t - r/c$ if $v \ll c$.

*Remark* 1. $t_e$ is the time of emission for a signal received at time $t$.

To find the power radiated, we will need the following quantity:

$$\begin{aligned}\underline{\int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)} &\approx \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_e) \\ &= \frac{d}{dt} e \int d\boldsymbol{y}\, \boldsymbol{v}(t_e)\, \delta(\boldsymbol{y} - \boldsymbol{R}(t_e)) \\ &= e \left.\frac{d\boldsymbol{v}}{dt}\right|_{t=t_e} \\ &= \underline{e\dot{\boldsymbol{v}}(t_e)}.\end{aligned}$$

Inserting into $\left(\hat{\boldsymbol{x}} \times \int d\boldsymbol{y}\, \partial_t \boldsymbol{j}(\boldsymbol{y}, t_r)\right)^2$ yields (disregarding $e$):

$$\begin{aligned}(\hat{\boldsymbol{x}} \times \dot{\boldsymbol{v}}(t_e))^2 &= \varepsilon_{ijk}\hat{x}_j \dot{v}_k \varepsilon_{ilm}\hat{x}_l \dot{v}_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\, \hat{x}_j \hat{x}_l \dot{v}_k \dot{v}_m \\ &= \dot{\boldsymbol{v}}^2 - (\hat{\boldsymbol{x}} \cdot \dot{\boldsymbol{v}})^2\end{aligned}$$

$$\Longrightarrow \underline{\frac{d\mathscr{P}}{d\Omega} = \frac{e^2}{4\pi c^3}\left[(\dot{\boldsymbol{v}}(t_e))^2 - (\hat{\boldsymbol{x}} \cdot \dot{\boldsymbol{v}}(t_e))^2\right]}.$$

Let $\theta$ be the angle between the acceleration at time $t_e$ and the radius vector to the observer.

$$\Longrightarrow \hat{\boldsymbol{x}} \cdot \dot{\boldsymbol{v}}(t_e) = \dot{\boldsymbol{v}}^2 \cos^2\theta$$

$$\Longrightarrow \boxed{\frac{d\mathscr{P}}{d\Omega} = \frac{e^2}{4\pi c^3}(\dot{\boldsymbol{v}}(t_e))^2 \sin^2[\theta(t_e)]}.$$

**Proposition 1. Larmor formula.** *The total power radiated by the accelerated charge is*

$$\boxed{\mathscr{P} = \frac{2e^2}{3c^3}\dot{\boldsymbol{v}}^2} \quad (\text{for } v \ll c)\,.$$

*This is called the* Larmor formula.

*Proof.* $\int d\Omega\, \sin^2\theta = 2\pi \int_{-1}^{1} d\eta\, \left(1-\eta^2\right) = \frac{8\pi}{3}$. □

*Remark* 2. This is called the *Larmor formula*, valid for non-relativistic particles.

*Remark* 3. This is the physics behind synchrotron radiation (see Problem #46).

*Remark* 4. This implies that a classical atom cannot be stable (see Problems #47, 48).

## 3.4 Dipole radiation

Consider a system of many slow moving ($v \ll c$) charges that is still small compared to $r$. We will still use the approximation $t_r \approx t_e$.

**Proposition 1.** *In this situation the radiated power per solid angle is*

$$\boxed{\frac{d\mathscr{P}}{d\Omega} = \frac{1}{4\pi c^3}\left(\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{d}}\right)^2},$$

*where $\boldsymbol{d}$ is the dipole moment of the charge distribution, given by*

$$\boldsymbol{d}(t) := \int d\boldsymbol{y}\, \boldsymbol{y}\rho(\boldsymbol{y}, t)$$

*and $\ddot{\boldsymbol{d}}$ is its second time derivative.*

*Remark* 1. With $\theta$ the angle between $\ddot{\boldsymbol{d}}$ and $\hat{\boldsymbol{x}}$, this becomes

$$\underline{\frac{d\mathscr{P}}{d\Omega} = \frac{1}{4\pi c^3}\sin^2\theta\left(\ddot{\boldsymbol{d}}\right)^2}.$$

*Remark* 2. For one point charge, $\rho(\boldsymbol{y}, t) = e\delta(\boldsymbol{y} - \boldsymbol{R}(t))$

$$\Longrightarrow \boldsymbol{d}(t) = \int d\boldsymbol{y}\, \boldsymbol{y} e\delta(\boldsymbol{y} - \boldsymbol{R}(t)) = e\boldsymbol{R}(t)$$

$$\Longrightarrow \ddot{\boldsymbol{d}} = e\dot{\boldsymbol{v}},$$

so it works for one particle.

**Lemma 1.**

$$\underline{\frac{d}{dt}\boldsymbol{d}(t) = \int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t)}$$

*Proof.* Charge conservation implies

$$\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0.$$

Integrating over space,

$$\begin{aligned}
\Longrightarrow 0 &= \int d\boldsymbol{y}\,\boldsymbol{y}\left[\nabla_{\boldsymbol{y}}\cdot\boldsymbol{j}\left(\boldsymbol{y},t\right)+\partial_t\rho\left(\boldsymbol{y},t\right)\right] \\
&\overset{1.}{=} \int d\boldsymbol{y}\,\left[\nabla_{\boldsymbol{y}}\left(\boldsymbol{y}\cdot\boldsymbol{j}\right)-\boldsymbol{j}+\boldsymbol{y}\partial_t\rho\left(\boldsymbol{y},t\right)\right] \\
&\overset{2.}{=} -\int d\boldsymbol{y}\,\boldsymbol{j}\left(\boldsymbol{y},t\right)+\frac{d}{dt}\int d\boldsymbol{y}\,\boldsymbol{y}\rho\left(\boldsymbol{y},t\right) \\
&= -\int d\boldsymbol{y}\,\boldsymbol{j}\left(\boldsymbol{y},t\right)+\frac{d}{dt}\boldsymbol{d}\left(t\right).
\end{aligned}$$

1. Product rule
2. $\int d\boldsymbol{y}\,\nabla_{\boldsymbol{y}}\left(\boldsymbol{y}\cdot\boldsymbol{j}\right)\to 0$ if $\boldsymbol{y}\cdot\boldsymbol{j}$ falls off fast enough.

□

*Proof. (of proposition)*
From § 3.2,

$$\begin{aligned}
4\pi c^3\frac{d\mathscr{P}}{d\Omega} &\overset{1.}{\approx} \left(\hat{\boldsymbol{x}}\times\int d\boldsymbol{y}\,\partial_t\boldsymbol{j}\left(\boldsymbol{y},t_e\right)\right)^2 \\
&= \left(\hat{\boldsymbol{x}}\times\frac{d}{dt}\int d\boldsymbol{y}\,\boldsymbol{j}\left(\boldsymbol{y},t_e\right)\right)^2 \\
&\overset{2.}{=} \left(\hat{\boldsymbol{x}}\times\frac{d}{dt}\left(\frac{d}{dt}\boldsymbol{d}\right)\right)^2 \\
&= \left(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{d}}\right)^2
\end{aligned}$$

1. Replacing $t_r$ with $t_e$.
2. From the lemma above.

□

*Remark* 3. This contribution is called *electric dipole radiation*.

Now we keep corrections to the approximation we made ($t_r \approx t_e$). From § 3.2, to find $\frac{d\mathscr{P}}{d\Omega}$ we need

$$\begin{aligned}
\int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t_r) &= \int d\boldsymbol{y}\, \boldsymbol{j}\Big(\boldsymbol{y}, \underbrace{t - \frac{r}{c}}_{t_e} + \frac{1}{c}\hat{\boldsymbol{x}} \cdot \boldsymbol{y} + \dots\Big) \\
&\overset{1.}{\approx} \int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t_e) + \frac{1}{c} \int d\boldsymbol{y}\; (\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\; \partial_t \boldsymbol{j}(\boldsymbol{y}, t)\big|_{t=t_e} \\
&\overset{2.}{=} \dot{\boldsymbol{d}}(t_e) + \frac{1}{c} \frac{d}{dt}\Big|_{t_e} \int d\boldsymbol{y} \Big[ \tfrac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\, \boldsymbol{j} + \tfrac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j})\, \boldsymbol{y} \\
&\qquad + \tfrac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\, \boldsymbol{j} - \tfrac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j})\, \boldsymbol{y} \Big] \\
&= \dot{\boldsymbol{d}}(t_e) - \frac{1}{2c} \frac{d}{dt}\Big|_{t_e} \int d\boldsymbol{y}\; [\boldsymbol{y}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j}) - \boldsymbol{j}(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})] + (\text{other term}) \\
&\overset{3.}{=} \dot{\boldsymbol{d}}(t_e) - \frac{1}{2c} \frac{d}{dt}\Big|_{t_e} \int d\boldsymbol{y}\, \hat{\boldsymbol{x}} \times (\boldsymbol{y} \times \boldsymbol{j}) + (\text{other term}) \\
&= \dot{\boldsymbol{d}}(t_e) - \hat{\boldsymbol{x}} \times \frac{d}{dt}\Big|_{t_e} \frac{1}{2c} \int d\boldsymbol{y}\, \boldsymbol{y} \times \boldsymbol{j}(\boldsymbol{y}, t) + (\text{other term}) \\
&\overset{4.}{=} \dot{\boldsymbol{d}}(t_e) - \hat{\boldsymbol{x}} \times \frac{d}{dt}\Big|_{t_e} \boldsymbol{m} + (\text{other term}) \\
&\overset{5.}{=} \dot{\boldsymbol{d}}(t_e) - \hat{\boldsymbol{x}} \times \dot{\boldsymbol{m}}(t_e) + (\text{other term})
\end{aligned}$$

1. Taylor expanded.
2. Used the lemma to replace $\int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t_e)$, split the integrand $(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\, \boldsymbol{j}$ into $\frac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\, \boldsymbol{j} + \frac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{y})\, \boldsymbol{j}$, and added to the integrand $0 = \frac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j})\, \boldsymbol{y} - \frac{1}{2}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j})\, \boldsymbol{y}$.
3. Used vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$.
4. By definition (Ch. 3 § 4.7), the magnetic dipole moment is $\boldsymbol{m}(t) := \frac{1}{2c} \int d\boldsymbol{y}\, \boldsymbol{y} \times \boldsymbol{j}(\boldsymbol{y}, t)$.
5. In this and the following sections, we use the notation $\frac{d}{dt}\big|_{t_e} \boldsymbol{m} =: \dot{\boldsymbol{m}}(t_e)$.

Therefore, in this approximation, the power per solid angle is

$$\boxed{\frac{d\mathscr{P}}{d\Omega} = \frac{1}{4\pi c^3} \left[\hat{\boldsymbol{x}} \times \left(\ddot{\boldsymbol{d}} - \hat{\boldsymbol{x}} \times \ddot{\boldsymbol{m}}\right)\right]^2},$$

with $\boldsymbol{d}$ and $\boldsymbol{m}$ the electric and magnetic (respectively) dipole moments of the source. Note that the "other term" in the proof that we have neglected is of the same order as the $\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{m}})$ term in $v/c$. We will discuss this later.

**Corollary 1.** *The total radiated power is*

$$\boxed{\mathscr{P} = \frac{2}{3c^3} \left[\left(\ddot{\boldsymbol{d}}\right)^2 + (\ddot{\boldsymbol{m}})^2\right]}.$$

*Proof.*

$$\begin{aligned}
4\pi c^3 \mathscr{P} &= 4\pi c^3 \int d\Omega \frac{d\mathscr{P}}{d\Omega} \\
&= \int d\Omega \left[\hat{\boldsymbol{x}} \times \left(\ddot{\boldsymbol{d}} - \hat{\boldsymbol{x}} \times \ddot{\boldsymbol{m}}\right)\right]^2 \\
&= \int d\Omega \left[\left(\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{d}}\right)^2 - 2\left(\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{d}}\right) \cdot (\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{m}})) + (\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{m}}))^2\right]
\end{aligned}$$

We consider these terms separately:

$$\underline{\int d\Omega\left(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{d}}\right)^2} \overset{1.}{=} 2\pi\int_{-1}^{1} d\eta\left(1-\eta^2\right)\ddot{\boldsymbol{d}}^2 = 4\pi\left(1-\frac{1}{3}\right)\ddot{\boldsymbol{d}}^2 = \underline{\frac{8\pi}{3}\ddot{\boldsymbol{d}}^2}$$

1. Choosing our coordinate system such that $\ddot{\boldsymbol{d}} \parallel \hat{\boldsymbol{z}}$ (using notation $\eta := \cos\theta$).

$$\underline{\int d\Omega\left(\hat{\boldsymbol{x}}\times\left(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}\right)\right)^2} \overset{1.}{=} \int d\Omega\left[\hat{\boldsymbol{x}}\left(\hat{\boldsymbol{x}}\cdot\ddot{\boldsymbol{m}}\right)-\ddot{\boldsymbol{m}}\right]^2 \overset{2.}{=} \int d\Omega\left[\eta^2\ddot{\boldsymbol{m}}^2-2\eta^2\ddot{\boldsymbol{m}}^2+\ddot{\boldsymbol{m}}^2\right]^2 = 2\pi\int_{-1}^{1} d\eta\left(1-\eta^2\right)\ddot{\boldsymbol{m}}^2 = \underline{\frac{8\pi}{3}\ddot{\boldsymbol{m}}^2}$$

1. Vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$.
2. Choosing our coordinate system such that $\ddot{\boldsymbol{m}} \parallel \hat{\boldsymbol{z}}$ (using notation $\eta := \cos\theta$).

$$\int d\Omega\left(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{d}}\right)\cdot\left(\hat{\boldsymbol{x}}\times\left(\hat{\boldsymbol{x}}\times\ddot{\boldsymbol{m}}\right)\right) = 0,$$

since the integral is odd in $\eta$ (to see this, let $\hat{\boldsymbol{x}} \to -\hat{\boldsymbol{x}}$). □

Now, what of the other term we've been ignoring?

*Remark* 4. The other term, given by

$$\cdots + \frac{1}{2c}\frac{d}{dt}\bigg|_{t_e}\int d\boldsymbol{y}\ \left[\boldsymbol{y}\left(\hat{\boldsymbol{x}}\cdot\boldsymbol{j}\right)+\boldsymbol{j}\left(\hat{\boldsymbol{x}}\cdot\boldsymbol{y}\right)\right]+\ldots$$

has the structure (for the $i^{\text{th}}$ component):

$$\int d\boldsymbol{y}\ \left(y_i j_j + j_i y_j\right) \overset{1.}{=} -\int d\boldsymbol{y}\, y_i y_j \nabla_{\boldsymbol{y}}\cdot\boldsymbol{j} \overset{2.}{=} \int d\boldsymbol{y}\, y_i y_j \partial_t\rho = \frac{d}{dt}\int d\boldsymbol{y}\, y_i y_j \rho\left(\boldsymbol{y},t\right) = \frac{d}{dt}Q_{ij}\left(t\right),$$

where $Q_{ij}(t) := \int d\boldsymbol{y}\, y_i y_j \rho(\boldsymbol{y},t)$ is the electric quadrupole moment of the charge distribution.

1. Integrating by parts "in reverse".
2. Continuity equation.

Thus, the contribution to $\mathscr{P}$ from this term is of order $\frac{1}{c^5}\dddot{Q}^2$.

*Remark* 5. The magnetic dipole moment has an extra $1/c$ in its definition. Thus, the magnetic dipole and electric quadrupole radiation terms are of the same order in $v/c$ (see Landau & Lifshitz 71).

# 4 Spectral distribution of radiated energy

In § 3 we calculated the *total* power radiated by a time-dependent source.

**question:** How is this energy distributed over different frequencies?

## 4.1 Retarded potentials in frequency space

From § 2.3,

$$\phi(\boldsymbol{x},t) = \int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\rho\left(\boldsymbol{y}, t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}\right).$$

Define a temporal Fourier transform (cf. § 2.2)[2]

$$\begin{aligned} f(\boldsymbol{x},\omega) &:= \int dt\, e^{i\omega t} f(\boldsymbol{x},t) \\ \Longrightarrow f(\boldsymbol{x},t) &= \int \frac{d\omega}{2\pi}\, e^{-i\omega t} f(\boldsymbol{x},\omega). \end{aligned}$$

$$\begin{aligned} \Longrightarrow \underline{\phi(\boldsymbol{x},\omega)} &= \int dt\, e^{i\omega t} \int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \underbrace{\int \frac{d\omega'}{2\pi}\, e^{-i\omega'(t-|\boldsymbol{x}-\boldsymbol{y}|/c)}\rho(\boldsymbol{y},\omega')}_{=\rho(\boldsymbol{y},t-|\boldsymbol{x}-\boldsymbol{y}|/c)} \\ &= \int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \int \frac{d\omega'}{2\pi}\, \rho(\boldsymbol{y},\omega') \underbrace{\int dt\, e^{i(\omega-\omega')t}}_{=2\pi\delta(\omega-\omega')} e^{i\omega'|\boldsymbol{x}-\boldsymbol{y}|/c} \\ &= \underline{\int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|/c}\rho(\boldsymbol{y},\omega)} \end{aligned}$$

**Proposition 1.** *The retarded potentials in frequency space are*

$$\boxed{\phi(\boldsymbol{x},\omega) = \int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|/c}\rho(\boldsymbol{y},\omega)}$$

*and, analogously,*

$$\boxed{\boldsymbol{A}(\boldsymbol{x},\omega) = \frac{1}{c}\int d\boldsymbol{y}\, \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|/c}\boldsymbol{j}(\boldsymbol{y},\omega)},$$

*where $\rho(\boldsymbol{y},\omega)$ and $\boldsymbol{j}(\boldsymbol{y},\omega)$ are the temporal Fourier transforms of $\rho(\boldsymbol{y},t)$ and $\boldsymbol{j}(\boldsymbol{y},t)$, respectively.*

*Proof.* (above) □

## 4.2 Asymptotic potentials and fields

For large distances $r := |\boldsymbol{x}|$ from the sources, the expansion from § 3.1 applies:

$$|\boldsymbol{x}-\boldsymbol{y}| \approx r - \hat{\boldsymbol{x}}\cdot\boldsymbol{y}$$

$$\begin{aligned} \Longrightarrow \phi(\boldsymbol{x},\omega) &= \int d\boldsymbol{y}\, \frac{1}{r}\left[1 + O\left(\frac{1}{r}\right)\right] e^{i\omega(r-\hat{\boldsymbol{x}}\cdot\boldsymbol{y}+\dots)/c}\rho(\boldsymbol{y},\omega) \\ &\approx \frac{1}{r}e^{i\omega r/c}\int d\boldsymbol{y}\, e^{-i\omega\hat{\boldsymbol{x}}\cdot\boldsymbol{y}/c}\rho(\boldsymbol{y},\omega) + O\left(1/r^2\right) \end{aligned}$$

[2] In this notation, the argument of the function indicates if it is a Fourier transform or not (the same symbol $f$ is used to refer to the Fourier transformed function and the original function).

**definition:** $\boxed{\boldsymbol{k} := \frac{\omega}{c}\hat{\boldsymbol{x}}}$ is called *wave vector.*

*Remark* 1. This is consistent with Ch. 4 § 1.5 Remark 1.

*Remark* 2. Far from the source, the wave fronts are approximately plane waves, so Ch. 4 applies.

$$\begin{aligned} \Longrightarrow \underline{\phi(\boldsymbol{x},\omega)} &\approx \frac{1}{r}e^{ikr}\int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\rho(\boldsymbol{y},\omega) \\ &= \underline{\frac{1}{r}e^{ikr}\rho(\boldsymbol{k},\omega)}, \end{aligned}$$

with $\rho(\boldsymbol{k},\omega)$ the spacial Fourier transform of $\rho(\boldsymbol{x},\omega)$, and $k := |\boldsymbol{k}|$. Note that $\frac{1}{r}e^{ikr}$ represents a spherical wave.

Analogously,

$$\underline{\boldsymbol{A}(\boldsymbol{x},\omega)} \approx \underline{\frac{1}{r}e^{ikr}\frac{1}{c}\boldsymbol{j}(\boldsymbol{k},\omega)}.$$

**Proposition 1.** *Far from the sources, the fields are*

$$\boxed{\boldsymbol{B}(\boldsymbol{x},\omega) \approx i\frac{\omega}{c}\frac{e^{i\omega r/c}}{r}\hat{\boldsymbol{x}} \times \frac{1}{c}\boldsymbol{j}(\boldsymbol{k},\omega)},$$

$$\boxed{\boldsymbol{E}(\boldsymbol{x},\omega) \approx -\hat{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x},\omega)}.$$

*Remark* 3. The expression for $\boldsymbol{E}$ in terms of $\boldsymbol{B}$ follows instantly from the proposition in § 3.1 (by taking Fourier transform).

*Proof. (of proposition)*

Taking the temporal Fourier transform of $\boldsymbol{B}(\boldsymbol{x},t) = \nabla \times \boldsymbol{A}(\boldsymbol{x},t)$ yields

$$\boldsymbol{B}(\boldsymbol{x},\omega) = \nabla \times \boldsymbol{A}(\boldsymbol{x},\omega)$$

$$\begin{aligned} \Longrightarrow \underline{B_l(\boldsymbol{x},\omega)} &= \varepsilon_{lmn}\partial_m A_n(\boldsymbol{x},\omega) \\ &= \varepsilon_{lmn}\partial_m\left(\frac{1}{r}e^{ikr}\right)\frac{1}{c}j_n(\boldsymbol{k},\omega) \\ &\overset{1.}{\approx} \varepsilon_{lmn}ik\frac{e^{ikr}}{r}\hat{x}_m\frac{1}{c}j_n(\boldsymbol{k},\omega) \\ &= \underline{ik\frac{e^{ikr}}{r}\left(\hat{\boldsymbol{x}} \times \frac{1}{c}\boldsymbol{j}(\boldsymbol{k},\omega)\right)_l}. \end{aligned}$$

1. The product rule yields two terms:

$$\underline{\partial_m\frac{1}{r}} = -\frac{\hat{x}_m}{r^2} + O\left(1/r^3\right) = \underline{O\left(1/r^2\right)} \ldots \text{ discard this term}$$

$$\underline{\frac{1}{r}\partial_m e^{ikr}} = \frac{1}{r}e^{ikr}ik\partial_m r = \frac{e^{ikr}}{r}ik\frac{1}{2r}2x_m = \underline{ik\frac{e^{ikr}}{r}\hat{x}_m} = O(1/r)$$

□

## 4.3 The spectral distribution of the radiated energy

**Theorem 1.** *The total radiated energy per solid angle $d\Omega$ and frequency $d\omega$ is*

$$\boxed{\frac{d^2U}{d\Omega\,d\omega}=\frac{\omega^2}{4\pi^2c^3}\left|\hat{\boldsymbol{x}}\times\boldsymbol{j}\left(\boldsymbol{k},\omega\right)\right|^2}.$$

*Remark* 1. Check a static source: $\boldsymbol{j}\left(\boldsymbol{k},t\right)=\boldsymbol{j}\left(\boldsymbol{k}\right)$

$$\Longrightarrow\ \boldsymbol{j}\left(\boldsymbol{k},\omega\right)\propto\delta\left(\omega\right)$$

$$\Longrightarrow\ \frac{d^2U}{d\Omega\,d\omega}=0$$

*Proof. (of theorem)*
The instantaneous flux of energy is given by the Poynting vector (Ch. 2 § 3.6):

$$\boldsymbol{P}\left(\boldsymbol{x},t\right):=\frac{c}{4\pi}\boldsymbol{E}\left(\boldsymbol{x},t\right)\times\boldsymbol{B}\left(\boldsymbol{x},t\right).$$

Then the total energy $U$ radiated into a solid angle is (see § 3.2):

$$\begin{aligned}
\frac{dU}{d\Omega} &= \int dt\,r^2\left(\hat{\boldsymbol{x}}\cdot\boldsymbol{P}\left(\boldsymbol{x},t\right)\right)\\
&= r^2\frac{c}{4\pi}\int dt\,\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{E}\left(\boldsymbol{x},t\right)\times\boldsymbol{B}\left(\boldsymbol{x},t\right)\right]\\
&= \frac{cr^2}{4\pi}\int dt\,\hat{\boldsymbol{x}}\cdot\left[\left(\int\frac{d\omega}{2\pi}\,e^{-i\omega t}\boldsymbol{E}\left(\boldsymbol{x},\omega\right)\right)\times\left(\int\frac{d\omega'}{2\pi}\,e^{-i\omega' t}\boldsymbol{B}\left(\boldsymbol{x},\omega'\right)\right)\right]\\
&\overset{1.}{=} \frac{cr^2}{4\pi}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\,\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{E}\left(\boldsymbol{x},\omega\right)\times\boldsymbol{B}\left(\boldsymbol{x},\omega'\right)\right]2\pi\delta\left(\omega+\omega'\right)\\
&= \frac{cr^2}{4\pi}\int\frac{d\omega}{2\pi}\,\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{E}\left(\boldsymbol{x},\omega\right)\times\boldsymbol{B}\left(\boldsymbol{x},-\omega\right)\right]\\
&\overset{2.}{=} -\frac{cr^2}{4\pi}\int\frac{d\omega}{2\pi}\,\hat{\boldsymbol{x}}\cdot\left[\left(\hat{\boldsymbol{x}}\times\boldsymbol{B}\left(\boldsymbol{x},\omega\right)\right)\times\boldsymbol{B}\left(\boldsymbol{x},\omega\right)^*\right]\\
&\overset{3.}{=} \frac{cr^2}{4\pi}\int\frac{d\omega}{2\pi}\,\left|\boldsymbol{B}\left(\boldsymbol{x},\omega\right)\right|^2\\
&\overset{4.}{=} \frac{cr^2}{4\pi^2}\int_0^\infty d\omega\,\left|\boldsymbol{B}\left(\boldsymbol{x},\omega\right)\right|^2\\
&\overset{5.}{=} \frac{1}{4\pi^2c^3}\int_0^\infty d\omega\,\omega^2\left|\hat{\boldsymbol{x}}\times\boldsymbol{j}\left(\boldsymbol{k},\omega\right)\right|^2
\end{aligned}$$

1. $\int dt\,e^{-i\left(\omega+\omega'\right)t}=2\pi\delta\left(\omega+\omega'\right)$.
2. Since $B_i\left(\boldsymbol{x},t\right)\in\mathbb{R}$, $\boldsymbol{B}\left(\boldsymbol{x},-\omega\right)=\boldsymbol{B}\left(\boldsymbol{x},\omega\right)^*$. Also, by § 4.2, $\boldsymbol{E}\left(\boldsymbol{x},\omega\right)\approx-\hat{\boldsymbol{x}}\times\boldsymbol{B}\left(\boldsymbol{x},\omega\right)$.
3. Since $\hat{\boldsymbol{x}}\perp\boldsymbol{B}$.
4. Since integrand is even in $\omega$.
5. From § 4.2 proposition, $\boldsymbol{B}\left(\boldsymbol{x},\omega\right)\approx i\frac{\omega}{c}\frac{e^{i\omega r/c}}{r}\hat{\boldsymbol{x}}\times\frac{1}{c}\boldsymbol{j}\left(\boldsymbol{k},\omega\right)$.

$$\Longrightarrow\ \frac{d^2U}{d\Omega\,d\omega}=\frac{\omega^2}{4\pi^2c^3}\left|\hat{\boldsymbol{x}}\times\boldsymbol{j}\left(\boldsymbol{k},\omega\right)\right|^2.$$

□

## 4.4 Spectral distribution for dipole radiation

From § 4.3, $\frac{d^2U}{d\Omega\, d\omega}$ is given by the Fourier transform of the current density:

$$\boldsymbol{j}(\boldsymbol{k},\omega), \text{ where } k = |\boldsymbol{k}| = \frac{\omega}{c} = \frac{2\pi}{\lambda},$$

with $\lambda$ the wavelength of the radiation.

Consider small sources in the limit that $|\boldsymbol{y}| \ll \lambda$.

**Example 1.** For an atom radiating visible light, we have

$$\begin{aligned} |\boldsymbol{y}| &\lesssim \text{a few Å} \\ \lambda &\approx \text{thousands of Å} \end{aligned}$$

In this limit,

$$\begin{aligned} \underline{\boldsymbol{j}(\boldsymbol{k},\omega)} &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \int dt\, e^{i\omega t} \boldsymbol{j}(\boldsymbol{y},t) \\ &\overset{1.}{=} \int d\boldsymbol{y}\, [1 - i\boldsymbol{k}\cdot\boldsymbol{y} + \ldots] \int dt\, e^{i\omega t} \boldsymbol{j}(\boldsymbol{y},t) \\ &\overset{2.}{=} \int dt\, e^{i\omega t} \int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y},t) + O(a/\lambda) \\ &\overset{3.}{=} \int dt\, e^{i\omega t} \frac{d}{dt} \boldsymbol{d}(t) + O(a/\lambda) \\ &= \underline{-i\omega \boldsymbol{d}(\omega) + O(a/\lambda)} \end{aligned}$$

1. Taylor expand $e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}$.
2. Define $a := |\boldsymbol{y}|$.
3. By § 3.4 lemma, $\frac{d}{dt}\boldsymbol{d}(t) = \int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y},t)$.

**Proposition 1.** *If $a$ is the linear dimension of the source, and $\lambda$ the wavelength of the radiation, then to lowest order in $a/\lambda \ll 1$ the energy radiated per unit solid angle and unit frequency is given by the* Larmor formula*:*

$$\boxed{\frac{d^2U}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} \sin^2\theta \left|\dot{\boldsymbol{d}}(\omega)\right|^2},$$

*where $\theta$ is the angle between $\boldsymbol{d}$, $\hat{\boldsymbol{x}}$*

$$\theta = \sphericalangle(\boldsymbol{d}, \hat{\boldsymbol{x}})$$

*and $\dot{\boldsymbol{d}}(\omega)$ is the Fourier transform of $\dot{\boldsymbol{d}}(t)$. That is,*

$$\dot{\boldsymbol{d}}(\omega) := -i\omega \boldsymbol{d}(\omega) = \mathcal{F}_t\left[\dot{\boldsymbol{d}}(t)\right](\omega).$$

*Proof.* In the dipole approximation, $\boldsymbol{d} \parallel \boldsymbol{j} \implies |\hat{\boldsymbol{x}} \times \boldsymbol{j}|^2 = \sin^2\theta\, |\boldsymbol{j}|^2$. □

**Corollary 1.** *The total energy per unit frequency is*

$$\boxed{\frac{dU}{d\omega} = \frac{2}{3}\frac{\omega^2}{\pi c^3} \left|\dot{\boldsymbol{d}}(\omega)\right|^2}.$$

*Proof.* $\int d\Omega\, \sin^2\theta = 2\pi \int_{-1}^{1} d\eta\, \left(1-\eta^2\right) = 4\pi\left(\frac{2}{3}\right) = \frac{8\pi}{3}$. □

**Example 2.** Consider a point charge $e$ on trajectory $\boldsymbol{y}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{y}}(t) \ll c$.

$$\implies \boldsymbol{j}(\boldsymbol{x},t) = e\boldsymbol{v}(t)\,\delta(\boldsymbol{x}-\boldsymbol{y}(t))$$

$$\implies \dot{\boldsymbol{d}}(t) = \int d\boldsymbol{x}\,\boldsymbol{j}(\boldsymbol{x},t) = e\boldsymbol{v}(t)$$

$$\implies \dot{\boldsymbol{d}}(\omega) = \mathcal{F}[e\boldsymbol{v}(t)](\omega) = e\boldsymbol{v}(\omega)$$

$$\begin{aligned}\implies \underline{\frac{dU}{d\omega}} &= \frac{2}{3}\frac{\omega^2 e^2}{\pi c^3}|\boldsymbol{v}(\omega)|^2 \\ &= \underline{\frac{2}{3}\frac{e^2}{\pi c^3}|\dot{\boldsymbol{v}}(\omega)|^2}\end{aligned}$$

$dU/d\omega$ is given by the Fourier transform of the acceleration.

*Remark* 1. This is consistent with the Larmor formula from § 3.3 (see Problem #52).

**Example 3.** Consider a slowly moving charge ($v \ll c$) on a circle.
$\implies$ $\dot{\boldsymbol{v}}$ is purely radial
$\implies$ power is maximal in the direction perpendicular to $\dot{\boldsymbol{v}}$ $\left(\theta = \pm\frac{\pi}{2}\right)$
$\implies$ *no* radiation emitted in direction of $\dot{\boldsymbol{v}}$ ($\theta = 0$)
$\implies$ in the orbital plane, the radiation has a butterfly shape
$\implies$ in 3-D, it has the shape of a torus

## 4.5 Example: radiation by a damped harmonic oscillator

Consider a charge $e$ in a harmonic potential (oscillator frequency $\omega_0$) with damping constant $\gamma$.

**equation of motion:**

$$\boxed{\ddot{y} = -\omega_0^2 y - \gamma\dot{y}} \qquad (*)$$

*Remark* 1. We think of the damping as due to the radiation emitted.

*Remark* 2. This is a simple model for an electron in a classical atom.

**initial conditions:**

$$y(t=0) = a, \quad \dot{y}(t=0) = 0.$$

**Lemma 1.** *For weak damping ($\gamma \ll \omega_0$), the solution of ($*$) is*

$$\boxed{y(t) \approx a\cos(\omega_0 t)\, e^{-\gamma t/2}} \qquad (t > 0)\,.$$

*Proof.* See Problem #53. □

$$\Longrightarrow \dot{y}(t) = -a\omega_0 \sin(\omega_0 t)\, e^{-\gamma t/2}\left[1 + O(\gamma/\omega_0)\right] =: v(t)$$

$$\begin{aligned}
\Longrightarrow v(\omega) &\approx -a\omega_0 \int_0^\infty dt\, e^{i\omega t} \sin(\omega_0 t)\, e^{-\gamma t/2} \\
&= -\frac{a\omega_0}{2i} \int_0^\infty dt\, e^{i\omega t}\left[e^{i\omega_0 t - \gamma t/2} - e^{-i\omega_0 t - \gamma t/2}\right] \\
&= -\frac{a\omega_0}{2i}\left[\frac{-1}{i(\omega+\omega_0) - \gamma/2} - \frac{-1}{i(\omega-\omega_0) - \gamma/2}\right] \\
&= \frac{a\omega_0}{2}\left[\frac{1}{\omega - \omega_0 + i\gamma/2} - \frac{1}{\omega+\omega_0+i\gamma/2}\right]
\end{aligned}$$

Let $\omega > 0$ (discussion for $\omega < 0$ is analogous). Then $v(\omega)$ is dominated by the first term when $\omega \approx \omega_0$.

$$\Longrightarrow |v(\omega)|^2 \approx \frac{a^2\omega_0^2}{4} \frac{1}{(\omega-\omega_0)^2 + \gamma^2/4}$$

$$\begin{aligned}
\Longrightarrow \underline{\frac{dU}{d\omega}} &= \frac{2e^2}{3\pi c^3}\left|\dot{\boldsymbol{v}}(\omega)\right|^2 \\
&= \frac{2e^2}{3\pi c^3}\frac{a^2\omega_0^2}{4}\frac{\omega^2}{(\omega-\omega_0)^2+\gamma^2/4} \\
&\approx \underline{\frac{2e^2}{3\pi c^3}\frac{a^2\omega_0^4}{4}\frac{1}{(\omega-\omega_0)^2+\gamma^2/4}} \quad (\omega \approx \omega_0).
\end{aligned}$$

This is sometimes called *susceptibility of oscillator.*

**discussion (1):** Spectrum is a Lorentzian centered on $\omega_0$ with width $\gamma$.

**discussion (2):** Total energy radiated:

$$\begin{aligned}
\underline{U} &= 2\int_0^\infty d\omega\, \frac{dU}{d\omega} \\
&\approx 2\frac{e^2a^2\omega_0^4}{6\pi c^3}\int_0^\infty d\omega\, \frac{1}{(\omega-\omega_0)^2+\gamma^2/4} \\
&\overset{1.}{=} \frac{e^2a^2\omega_0^4}{3\pi c^3}\int_{-\omega_0}^\infty d\omega\, \frac{1}{\omega^2+\gamma^2/4} \\
&\overset{2.}{=} \frac{e^2a^2\omega_0^4}{3\pi c^3}\left(\frac{2}{\gamma}\right)\int_{-\frac{2}{\gamma}\omega_0}^\infty dx\, \frac{1}{x^2+1} \\
&\overset{3.}{\approx} \frac{2e^2a^2\omega_0^4}{3\pi c^3\gamma}\underbrace{\int_{-\infty}^\infty dx\, \frac{1}{x^2+1}}_{\pi} \\
&= \underline{\frac{2e^2a^2\omega_0^4}{3c^3\gamma}}.
\end{aligned}$$

Let's compare with initial oscillator energy:

$$U_{\text{osc}}^{t=0} = \frac{m}{2}\omega_0^2 a^2$$

$$\begin{aligned}
\Longrightarrow U &= \frac{U_{\text{osc}}^{t=0}}{\frac{m}{2}\omega_0^2a^2}\frac{2e^2a^2\omega_0^4}{3c^3\gamma} \\
&= U_{\text{osc}}^{t=0}\frac{4e^2\omega_0^2}{3mc^3\gamma}.
\end{aligned}$$

Now, assuming the oscillator energy has totally gone into $U$, $\Longrightarrow U = U_{\text{osc}}^{t=0}$

$$\Longrightarrow \boxed{\gamma = \frac{4}{3}\frac{e^2\omega_0^2}{mc^3}}.$$

**discussion (3):** Compare this result with Problem #47:

$$\Longrightarrow U_{\text{osc}} = U_{\text{osc}}^{t=0} e^{-t/\tau}$$

where we found $\tau = 2/\gamma$. So the two approaches are consistent.

**discussion (4):** See Problem #53 for a more thorough discussion of the approximations made above.

# 5 Cherenkov radiation

## 5.1 The time-Wigner function, and the macroscopic power spectrum

From § 4.3, the spectral distribution of radiation from a time-dependent current density:

$$\begin{aligned}
\frac{d^2U}{d\Omega\, d\omega} &= \frac{\omega^2}{4\pi^2c^3}\left|\hat{\boldsymbol{x}} \times \boldsymbol{j}(\boldsymbol{k},\omega)\right|^2 \\
&= \frac{\omega^2}{4\pi^2c^3}\left(\hat{\boldsymbol{x}} \times \int dt\, e^{+i\omega t}\boldsymbol{j}(\boldsymbol{k},t)\right)\cdot\left(\hat{\boldsymbol{x}} \times \int dt'\, e^{-i\omega t'}\boldsymbol{j}(\boldsymbol{k},t')^*\right) \\
&= \frac{\omega^2}{4\pi^2c^3}\varepsilon_{ijk}\hat{x}_j\varepsilon_{ilm}\hat{x}_l \int dt \int dt'\, e^{i\omega(t-t')} j_k(\boldsymbol{k},t)\, j_m(\boldsymbol{k},t')^*
\end{aligned}$$

We can rewrite the integrals using the substitutions

$$\begin{aligned}
t &= T + \frac{\tau}{2} \\
t' &= T - \frac{\tau}{2}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \int dt \int dt'\, e^{i\omega(t-t')} j_k(\boldsymbol{k},t)\, j_m(\boldsymbol{k},t')^* &= \int dT \int d\tau\, e^{i\omega\tau} j_k\left(\boldsymbol{k}, T+\frac{\tau}{2}\right) j_m\left(\boldsymbol{k}, T-\frac{\tau}{2}\right)^* \\
&= \int dT \int d\tau\, e^{i\omega\tau} W_{km}(\boldsymbol{k};T,\tau),
\end{aligned}$$

where

$$\underline{W_{km} := j_k\left(\boldsymbol{k}, T+\frac{\tau}{2}\right) j_m\left(\boldsymbol{k}, T-\frac{\tau}{2}\right)^*}.$$

*Remark* 1. $W_{km}$ is an example of what is called a Wigner function (in our case a time-Wigner function). It separates the two times into *average time* (or *macroscopic time*) $T$ and *relative time* (or *microscopic time*) $\tau$.

*Remark* 2. Only relative times $|\tau| \lesssim 1/\omega$ will appreciably contribute to the $\tau-$integral, whereas all times $T$ during which the source is active contribute to the $T-$integral.

*Remark* 3. This makes sense if the two time-scales are well separated. E.g., a laser pulse of duration $T \gg 1/\omega$.

**Definition 1.** The spectral distribution at time $T$ is

$$\boxed{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega} := \frac{\omega^2}{4\pi^2c^3}\varepsilon_{ijk}\hat{x}_j\varepsilon_{ilm}\hat{x}_l \int d\tau\, e^{i\omega\tau} W_{km}(\boldsymbol{k};T,\tau)},$$

called the *macroscopic power spectrum.*

*Remark* 4. We recover $\frac{d^2U}{d\Omega\, d\omega}$ as $\frac{d^2U}{d\Omega\, d\omega} = \int dT\, \frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}$.

## 5.2 Cherenkov radiation

Consider a point particle as in § 3.3:

$$\boldsymbol{j}(\boldsymbol{y},t) = e\boldsymbol{v}(t)\,\delta(\boldsymbol{y}-\boldsymbol{R}(t)),\ \text{where } \boldsymbol{v}(t) := \dot{\boldsymbol{R}}(t).$$

We specialize to uniform motion along a straight line:

$$\boldsymbol{R}(t) = \boldsymbol{v}t, \quad \boldsymbol{v}(t) = \boldsymbol{v} = \text{const.}$$

*Remark* 1. We know that in vacuum this does not result in radiation.

$$\begin{aligned} \Longrightarrow \boldsymbol{j}(\boldsymbol{k},t) &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} e\boldsymbol{v}\delta(\boldsymbol{y}-\boldsymbol{v}t) \\ &= e\boldsymbol{v}e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t} \\ &= e\boldsymbol{v}e^{-i\hat{\boldsymbol{x}}\cdot\boldsymbol{v}\frac{\omega}{c}t} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \underline{W_{km}(\boldsymbol{k};T,\tau)} &= j_k\left(\boldsymbol{k},T+\frac{\tau}{2}\right) j_m\left(\boldsymbol{k},T-\frac{\tau}{2}\right)^* \\ &= e^2 v_k v_m e^{-i\hat{\boldsymbol{x}}\cdot\boldsymbol{v}\frac{\omega}{c}\left(T+\frac{\tau}{2}\right)} e^{+i\hat{\boldsymbol{x}}\cdot\boldsymbol{v}\frac{\omega}{c}\left(T-\frac{\tau}{2}\right)} \\ &= \underline{e^2 v_k v_m e^{-i\hat{\boldsymbol{x}}\cdot\boldsymbol{v}\frac{\omega}{c}\tau}}. \end{aligned}$$

*Remark* 2. The Wigner function is independent of $T$ here, as expected from uniform motion.

$$\begin{aligned} \Longrightarrow \underline{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}} &:= \frac{\omega^2}{4\pi^2c^3}\varepsilon_{ijk}\hat{x}_j\varepsilon_{ilm}\hat{x}_l \int d\tau\, e^{i\omega\tau} W_{km}(\boldsymbol{k};T,\tau) \\ &= \frac{\omega^2e^2}{4\pi^2c^3}\varepsilon_{ijk}\varepsilon_{ilm}\hat{x}_j\hat{x}_l v_k v_m \int d\tau\, e^{i\omega\tau}e^{-i\hat{\boldsymbol{x}}\cdot\boldsymbol{v}\frac{\omega}{c}\tau} \\ &\overset{1.}{=} \frac{\omega^2e^2}{4\pi^2c^3} v^2\sin^2\theta \int d\tau\, e^{i\omega\left(1-\frac{v}{c}\cos\theta\right)\tau} \\ &\overset{2.}{=} \frac{\omega^2e^2}{4\pi^2c}\left(\frac{v}{c}\right)^2 \sin^2\theta\,\delta\left(\omega\left(1-\frac{v}{c}\cos\theta\right)\right) \\ &= \underline{\frac{|\omega|\,e^2}{4\pi^2c}\left(\frac{v}{c}\right)^2 \sin^2\theta\,\delta\left(1-\frac{v}{c}\cos\theta\right)} \end{aligned}$$

1. Defining $\theta$ to be the angle between $\hat{\boldsymbol{x}}$ and $\boldsymbol{v}$:

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{ilm}\hat{x}_j\hat{x}_l v_k v_m &= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\,\hat{x}_j\hat{x}_l v_k v_m \\ &= \boldsymbol{v}^2 - (\hat{\boldsymbol{x}}\cdot\boldsymbol{v})^2 \\ &= v^2\sin^2\theta \end{aligned}$$

2. $\int d\tau\, e^{i\omega\left(1-\frac{v}{c}\cos\theta\right)\tau} = \delta\left(\omega\left(1-\frac{v}{c}\cos\theta\right)\right)$

*Remark* 3. If $v/c < 1$, $|\cos\theta| < 1 \implies 1+\frac{v}{c}\cos\theta > 0$
$\Longrightarrow$ no radiation, in agreement with § 3.3 unless $v > c$ ("tachyonic particle").

*Remark* 4. In matter, $c \to c/n$, with $n$ the index of refraction

$$\Longrightarrow \frac{v}{c} \to n\frac{v}{c} \implies n\frac{v}{c} > 1 \text{ is possible!}$$

*Remark* 5. Strictly speaking, this requires a theory for electromagnetic radiation in matter. Here we assume $c \to c/n$ and $e^2 \to e^2/n^2$ suffices to catch the main effects (the charge is screened; $F_{\text{coulomb}} = e^2/r \to e^2/(\varepsilon r)$ where $n = \sqrt{\varepsilon}$).

Also keep in mind we are applying a nonrelativistic approximation to a situation in which $v/c$ is no longer small (see Problem #54).

*Remark* 6. $n$ is frequency dependent ($n = n(\omega)$)

$$\begin{aligned} \Longrightarrow \underline{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}} &= \frac{|\omega|\, e^2/n^2}{4\pi^2 c/n}\left(n\frac{v}{c}\right)^2 \sin^2\theta\delta\left(1 - n\frac{v}{c}\cos\theta\right) \\ &= \underline{\frac{|\omega|\, e^2}{4\pi^2 c n(\omega)}\left(n(\omega)\frac{v}{c}\right)^2 \sin^2\theta\delta\left(1 - n(\omega)\frac{v}{c}\cos\theta\right)}. \end{aligned}$$

**conclusion:** a particle moving in a medium faster than the speed of light in that medium emits radiation (*Cherenkov radiation*) on a cone with angle $\theta$ where

$$\cos\theta = \frac{c}{vn(\omega)}.$$

**Proposition 1.** *The total power emitted is*

$$\boxed{\frac{d\mathscr{P}}{d\omega} = |\omega|\frac{e^2 v}{2\pi c^2}\left(1 - \frac{c^2}{n^2(\omega)\, v^2}\right)}.$$

*Proof.*

$$\begin{aligned} \frac{d\mathscr{P}}{d\omega} &= \int d\Omega\, \frac{d^2\mathscr{P}}{d\Omega\, d\omega} \\ &= \frac{|\omega|\, e^2}{4\pi^2 cn}\left(n\frac{v}{c}\right)^2 (2\pi)\int_{-1}^{1} d\eta\, \left(1-\eta^2\right)\delta\left(1 - n\frac{v}{c}\eta\right) \\ &= \frac{|\omega|\, e^2}{2\pi cn}\left(n\frac{v}{c}\right)^2 \frac{c}{nv}\int_{-1}^{1} d\eta\, \left(1-\eta^2\right)\delta\left(\eta - \frac{c}{vn}\right) \\ &= \frac{|\omega|\, e^2 v}{2\pi c^2}\left(1 - \left(\frac{c}{nv}\right)^2\right). \end{aligned}$$

□

*Remark* 7. This is nonzero only for the range of frequencies (if they exist) such that $vn(\omega) > c$.
$\Longrightarrow$ total radiated power, $\mathscr{P} = \int d\omega\, \frac{d\mathscr{P}}{d\omega}$ is finite.

*Remark* 8. This is radiated energy per time and frequency, whereas a Cherenkov radiation detector observes the energy radiated per distance traveled by the particle. Now,

$$\begin{aligned} \mathscr{P} = \frac{dE}{dt} &= \int d\omega\, \frac{d\mathscr{P}}{d\omega} \\ &= \frac{e^2 v}{2\pi c^2} 2\int_0^\infty d\omega\, \omega\left(1 - \frac{c^2}{n^2(\omega)\, v^2}\right)\Theta\left(\frac{c^2}{n^2(\omega)\, v^2} < 1\right), \end{aligned}$$

where $\Theta\left(\frac{c^2}{n^2(\omega)v^2} < 1\right)$ is the theta function.

$$\Longrightarrow \underline{\frac{dE}{dx}} = \frac{dE}{dt}\frac{dt}{dx} = \underline{\frac{e^2}{\pi c^2}\int_0^\infty d\omega\, \omega\left(1 - \frac{c^2}{n^2(\omega)\, v^2}\right)\Theta\left(\frac{c^2}{n^2(\omega)\, v^2} < 1\right)}.$$

*Remark* 9. Each photon has energy $E = \hbar\omega$
$\implies$ the number of photons per distance and frequency is

$$\underline{\frac{d^2N}{dx\,d\omega}} = \frac{e^2}{\hbar\pi c^2}\left(1 - \frac{c^2}{n^2(\omega)\,v^2}\right)\Theta\left(\frac{c^2}{n^2(\omega)\,v^2} < 1\right)$$
$$= \underline{\frac{\alpha}{\pi c}\left(1 - \frac{c^2}{n^2(\omega)\,v^2}\right)\Theta\left(\frac{c^2}{n^2(\omega)\,v^2} < 1\right)},$$

with $\alpha := \frac{e^2}{\hbar c} \approx \frac{1}{137}$ the *fine structure constant.*

# 6 Synchrotron radiation

**idea:** Discuss motion of charged particle in a homogeneous $B-$field, as in Problem #46, but

- do it relativistically
- discuss the power spectrum

## 6.1 Relativistic motion of a charged particle in a $B-$field

From PHYS611,

$$\boxed{\frac{d\boldsymbol{p}}{dt} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B}}, \qquad (*)$$

with $\boldsymbol{p} = \gamma m\boldsymbol{v}$ the momentum $\left(\gamma := 1/\sqrt{1-(v/c)^2}\right)$.

*Remark* 1. $(*)$ holds for relativistic motion.

*Remark* 2. Force is purely transverse $\implies E = \gamma mc^2 = \text{const.}$, and $\boldsymbol{p} = \frac{E}{c^2}\boldsymbol{v}$ with $E$ the particle's energy.
$\implies$ $(*)$ can be written

$$\frac{E}{c^2}\frac{d\boldsymbol{v}}{dt} = \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B} \implies \underline{\frac{d\boldsymbol{v}}{dt}} = \frac{ec}{E}\boldsymbol{v} \times \boldsymbol{B} = \underline{-\frac{ec}{E}\boldsymbol{B} \times \boldsymbol{v}}.$$

**Definition 1.** ***Larmor frequency.***

$$\omega_0 := \frac{|e|\,cB}{E},$$

is called *Larmor frequency.*

*Remark* 3. In nonrelativistic limit, $\omega_0 \approx \frac{|e|cB}{mc^2} = \frac{|e|B}{mc}$, called *cyclotron frequency.*

**initial condition:** $\boldsymbol{v} \perp \boldsymbol{B} \implies \boldsymbol{v} \perp \boldsymbol{B}$ for all times.

**conclusion:** particle moves on a circle perpendicular to $B-$field of radius

$$\boxed{R = \frac{v}{\omega_0} = \frac{v}{c}\frac{E}{|e|\,B}},$$

and the momentum is related to the radius by

$$\boxed{p = \frac{E}{c^2}v = \frac{1}{c}\,|e|\,BR}.$$

## 6.2 The power spectrum of synchrotron radiation

Consider motion in the $x - y$ plane with an observer at point $\boldsymbol{x}$ and $\theta = \sphericalangle(\boldsymbol{x}, \hat{\boldsymbol{z}})$. Choose coordinate system such that $\boldsymbol{x} = (x, 0, z)$

$$\Longrightarrow \boxed{\hat{\boldsymbol{x}} = (\sin\theta, 0, \cos\theta)}.$$

and initial conditions such that $\boldsymbol{y}(t) = R(\cos\omega_0 t, \sin\omega_0 t, 0)$

$$\Longrightarrow \boxed{\boldsymbol{v}(t) = v(-\sin\omega_0 t, \cos\omega_0 t, 0)} \text{ with } v = \omega_0 R.$$

**current density:** $\boldsymbol{j}(\boldsymbol{y}, t) = e\boldsymbol{v}(t)\,\delta(\boldsymbol{y} - \boldsymbol{y}(t))$

$$\begin{aligned}
\Longrightarrow \underline{\boldsymbol{j}(\boldsymbol{k}, t)} &= \int d\boldsymbol{y}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} e\boldsymbol{v}(t)\,\delta(\boldsymbol{y} - \boldsymbol{y}(t)) \\
&= e\boldsymbol{v}(t)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}(t)} \\
&= \underline{e\boldsymbol{v}(t)\, e^{-i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}(t)}}.
\end{aligned}$$

**charge density:** $\rho(\boldsymbol{y}, t) = e\delta(\boldsymbol{y} - \boldsymbol{y}(t))$

$$\begin{aligned}
\Longrightarrow \underline{\rho(\boldsymbol{k}, t)} &= e e^{-i\boldsymbol{k}\cdot\boldsymbol{y}(t)} \\
&= \underline{e e^{-i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}(t)}}
\end{aligned}$$

**Lemma 1.** *The power spectrum from § 5.1 can be written*

$$\boxed{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3}\int d\tau\, e^{i\omega\tau}\left[\boldsymbol{j}\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\cdot\boldsymbol{j}\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^* - c^2\rho\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\rho\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^*\right]}$$

*Proof.* From § 5.1, the integrand (ignoring $e^{i\omega\tau}$ factor and coefficients) is

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{ilm}\hat{x}_j\hat{x}_l W_{km}(\boldsymbol{k}; T, \tau) &\overset{1.}{=} \varepsilon_{ijk}\hat{x}_j\varepsilon_{ilm}\hat{x}_l j_k\left(\boldsymbol{k}, T + \frac{\tau}{2}\right) j_m\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^* \\
&\overset{2.}{=} \boldsymbol{j}\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\cdot\boldsymbol{j}\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^* - \left(\hat{\boldsymbol{x}}\cdot\boldsymbol{j}\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\right)\left(\hat{\boldsymbol{x}}\cdot\boldsymbol{j}\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^*\right) \\
&\overset{3.}{=} \boldsymbol{j}\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\cdot\boldsymbol{j}\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^* - \left(c\rho\left(\boldsymbol{k}, T + \frac{\tau}{2}\right)\right)\left(c\rho\left(\boldsymbol{k}, T - \frac{\tau}{2}\right)^*\right)
\end{aligned}$$

1. By definition (see § 5.1).
2. From § 5.2 Remark 2.
3. At asymptotic distances away from source, $\hat{\boldsymbol{x}} \approx \hat{\boldsymbol{k}}$. Using $|\boldsymbol{k}| = \frac{\omega}{c}$ and continuity eq. yields

$$\begin{aligned}
\partial_t\rho(\boldsymbol{x}, t) &= -\nabla\cdot\boldsymbol{j}(\boldsymbol{x}, t) \\
\xrightarrow{\mathcal{F}} i\omega\rho(\boldsymbol{k}, \omega) &= i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{j}(\boldsymbol{k}, \omega) \\
\xrightarrow{\mathcal{F}^{-1}} c\rho(\boldsymbol{k}, t) &= \hat{\boldsymbol{x}}\cdot\boldsymbol{j}(\boldsymbol{k}, t)
\end{aligned}$$

□

**Lemma 2.**

$$\boxed{\boldsymbol{v}\left(T + \frac{\tau}{2}\right)\cdot\boldsymbol{v}\left(T - \frac{\tau}{2}\right) = v^2\cos\omega_0\tau}$$

*Proof.*

$$
\begin{aligned}
\frac{1}{v^2}\boldsymbol{v}\left(T+\frac{\tau}{2}\right)\cdot\boldsymbol{v}\left(T-\frac{\tau}{2}\right) &\overset{1.}{=} \sin\left(\omega_0\left(T+\frac{\tau}{2}\right)\right)\sin\left(\omega_0\left(T-\frac{\tau}{2}\right)\right)+\cos\left(\omega_0\left(T+\frac{\tau}{2}\right)\right)\cos\left(\omega_0\left(T-\frac{\tau}{2}\right)\right)\\
&\overset{2.}{=} \cos\left(\omega_0\left(T+\frac{\tau}{2}\right)-\omega_0\left(T-\frac{\tau}{2}\right)\right)\\
&= \cos\omega_0\tau
\end{aligned}
$$

1. $\frac{\boldsymbol{v}(t)}{v}=(-\sin\omega_0 t,\cos\omega_0 t,0)$
2. Angle difference formula.

$\square$

**Lemma 3.**

$$
\boxed{e^{\mp\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}\left(T\pm\frac{\tau}{2}\right)}=\sum_{m=-\infty}^{\infty}(\mp i)^m\, e^{\mp im\omega_0\left(T\pm\frac{\tau}{2}\right)}J_m\left(\frac{\omega}{c}R\sin\theta\right)},
$$

*with $J_m(x)$ a Bessel function of the first kind.*

*Proof.* The Bessel functions obey

$$
e^{iz\cos\varphi}=\sum_{m=-\infty}^{\infty}i^m e^{im\varphi}J_m(z)
$$

and $\hat{\boldsymbol{x}}\cdot\boldsymbol{y}(t)=R\sin\theta\cos\omega_0\tau$

$$
\begin{aligned}
\implies e^{\mp i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\boldsymbol{y}(t)} &= e^{\mp i\frac{\omega}{c}R\sin\theta\cos\omega_0 t}\\
&= \sum_{m=-\infty}^{\infty}(\mp i)^m\, e^{\mp im\omega_0 t}J_m\left(\frac{\omega}{c}R\sin\theta\right).
\end{aligned}
$$

$\square$

$$
\begin{aligned}
\implies \underline{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}} &\overset{1.}{=} \frac{\omega^2}{4\pi^2c^3}\int d\tau\, e^{i\omega\tau}\left[\boldsymbol{j}\left(\boldsymbol{k},T+\frac{\tau}{2}\right)\cdot\boldsymbol{j}\left(\boldsymbol{k},T-\frac{\tau}{2}\right)^*-c^2\rho\left(\boldsymbol{k},T+\frac{\tau}{2}\right)\rho\left(\boldsymbol{k},T-\frac{\tau}{2}\right)^*\right]\\
&= \frac{\omega^2e^2}{4\pi^2c^3}\int d\tau\, e^{i\omega\tau}\left[\boldsymbol{v}\left(T+\frac{\tau}{2}\right)\cdot\boldsymbol{v}\left(T+\frac{\tau}{2}\right)-c^2\right]e^{-i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right]}\\
&\overset{2.}{=} \frac{\omega^2e^2}{4\pi^2c}\int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau-1\right]\\
&\qquad \sum_{m=-\infty}^{\infty}(-i)^m\, e^{-im\omega_0\left(T+\frac{\tau}{2}\right)}J_m\left(\frac{\omega}{c}R\sin\theta\right)\sum_{n=-\infty}^{\infty}i^n e^{in\omega_0\left(T-\frac{\tau}{2}\right)}J_n\left(\frac{\omega}{c}R\sin\theta\right)\\
&= \underline{\frac{\omega^2e^2}{4\pi^2c}\sum_{m,n=-\infty}^{\infty}i^{n-m}e^{-i(m-n)\omega_0T}}\\
&\qquad \underline{\int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau-1\right]e^{-i(m+n)\omega_0\tau/2}J_m\left(\frac{\omega}{c}R\sin\theta\right)J_n\left(\frac{\omega}{c}R\sin\theta\right)}
\end{aligned}
$$

1. Lemma 1.
2. Lemmas 2, 3.

*Remark* 1. For the macroscopic power spectrum, we are not interested in how the emission varies on the microscopic time scale given by $1/\omega_0$.
$\Longrightarrow$ average over one oscillation period.

**Lemma 4.** *Let $\overline{f(T)}$ be a time average over one oscillation period. Then*

$$\boxed{\overline{e^{-i(m-n)T\omega_0}} = \delta_{mn}}.$$

*Proof.*

$$\begin{aligned}\overline{e^{-i(m-n)T\omega_0}} &= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dT\, e^{-i(m-n)\omega_0 T}\\ &= \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-i(m-n)x}\\ &= \delta_{mn}\end{aligned}$$

$\square$

$$\begin{aligned}\Longrightarrow \overline{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}} &= \frac{\omega^2 e^2}{4\pi^2 c}\sum_{m,n=-\infty}^{\infty} i^{n-m}\overline{e^{-i(m-n)\omega_0 T}}\\ &\quad \int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] e^{-i(m+n)\omega_0\tau/2} J_m\left(\frac{\omega}{c}R\sin\theta\right) J_n\left(\frac{\omega}{c}R\sin\theta\right)\\ &= \frac{\omega^2 e^2}{4\pi^2 c}\sum_{m=-\infty}^{\infty}\int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\frac{1}{2}\left(e^{i\omega_0\tau}+e^{-i\omega_0\tau}\right)-1\right] e^{-im\omega_0\tau}\left(J_m\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\\ &= \frac{\omega^2 e^2}{2\pi c}\sum_{m=-\infty}^{\infty}\left[\frac{v^2}{2c^2}\left(\delta\left(\omega-(m-1)\omega_0\right)+\delta\left(\omega-(m+1)\omega_0\right)\right)-\delta\left(\omega-m\omega_0\right)\right]\left(J_m\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\\ &\overset{1.}{=} \frac{\omega^2 e^2}{2\pi c}\sum_{m=-\infty}^{\infty}\\ &\quad \left[\frac{v^2}{2c^2}\left(\left(J_{m+1}\left(\frac{\omega}{c}R\sin\theta\right)\right)^2+\left(J_{m-1}\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\right)-\left(J_m\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\right]\delta\left(\omega-m\omega_0\right)\\ &\overset{2.}{=} \frac{\omega^2 e^2}{2\pi c}\left(\sum_{m=1}^{\infty}+\sum_{m=-1}^{-\infty}+\cancel{\sum_m\delta_{m0}}\right)[\cdots]\,\delta\left(\omega-m\omega_0\right)\\ &= \underline{\frac{\omega^2 e^2}{2\pi c}\sum_{m=1}^{\infty}}\\ &\quad \underline{\left[\frac{v^2}{2c^2}\left(\left(J_{m+1}\left(\frac{\omega}{c}R\sin\theta\right)\right)^2+\left(J_{m-1}\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\right)-\left(J_m\left(\frac{\omega}{c}R\sin\theta\right)\right)^2\right]\delta\left(\omega-m\omega_0\right)}\end{aligned}$$

$(*)$

1. Distributed the $J_m$ factor, shifted the sum indices so that $\delta\left(\omega-(m\pm1)\omega_0\right)\left(J_m\left(\frac{\omega}{c}R\sin\theta\right)\right)^2 \to \delta\left(\omega-m\omega_0\right)\left(J_{m\mp1}\left(\frac{\omega}{c}R\sin\theta\right)\right)^2$.
2. The summation can be split into three summations.
   (a) The $\sum\limits_m\delta_{m0}$ term does not contribute since $\omega^2\delta(\omega)=0$.
   (b) $J_{-m}(x)=(-)^m J_m(x)$, so the remaining two summations yield equivalent contributions.

*Remark* 2. The frequencies emitted are the Larmor frequency ($\omega_0$) and all of its harmonics.

**Theorem 1.** *The macroscopic power spectrum averaged over a microscopic period is*

$$\boxed{\overline{\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega}} = \sum_{m=1}^{\infty} \delta(\omega - m\omega_0)\frac{d\mathscr{P}_m}{d\Omega}},$$

*with the power radiated to the* $m^{\text{th}}$ *harmonic*

$$\boxed{\frac{d\mathscr{P}_m}{d\Omega} := \frac{\omega_0 e^2}{\pi R}\left(\frac{v}{c}\right)^3 m^2\left[\left(J'_m\left(m\frac{v}{c}\sin\theta\right)\right)^2 + \left(\frac{J_m\left(m\frac{v}{c}\sin\theta\right)}{\frac{v}{c}\tan\theta}\right)^2\right]}.$$

*Proof.* From ($*$), the argument of the Bessel functions is (applying the $\delta-$function)

$$x := \frac{\omega R}{c}\sin\theta = m\frac{\omega_0 R}{c}\sin\theta = m\frac{v}{c}\sin\theta$$

and the Bessel functions obey the recursion relations

$$\begin{aligned} J_{m-1}(x) - J_{m+1}(x) &= 2J'_m(x) \\ J_{m-1}(x) + J_{m+1}(x) &= \frac{2m}{x}J_m(x). \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - \frac{c^2}{v^2}J_m^2 &= \frac{1}{2}\left[\left(\frac{m}{x}J_m - J'_m\right)^2 + \left(\frac{m}{x}J_m + J'_m\right)^2\right] - \frac{m^2\sin^2\theta}{x^2}J_m^2 \\ &= \left(J'_m\right)^2 + \frac{m^2}{x^2}J_m^2 - \frac{m^2\sin^2\theta}{x^2}J_m^2 \\ &= \left(J'_m\right)^2 + \frac{m^2}{x^2}\left(1-\sin^2\theta\right)J_m^2 \\ &= \left(J'_m\right)^2 + \frac{c^2}{v^2}\underbrace{\frac{1-\sin^2\theta}{\sin^2\theta}}_{1/\tan^2\theta}J_m^2 \end{aligned}$$

□

**discussion (1):** Integration over $\Omega$ yields the total power radiated into the $m^{\text{th}}$ harmonic (see Problem #56):

$$\int d\Omega\,\frac{d\mathscr{P}_m}{d\Omega} = \boxed{\mathscr{P}_m = \frac{e^2}{R}m\omega_0\left[2\beta^2 J'_{2m}(2m\beta) - \left(1-\beta^2\right)\int_0^{2m\beta} dx\, J_{2m}(x)\right]}.$$

An analysis (Problem #56) shows that $\mathscr{P}_m$ peaks at

$$m = m_c \approx \gamma^3, \quad \gamma := \frac{1}{\sqrt{1-\beta^2}}.$$

$\Longrightarrow$ For relativistic electrons, power goes into a high harmonic.
$\Longrightarrow$ Synchrotrons are good *x-ray sources*.

**discussion (2):** In the orbital plane, $\theta = \pi/2$, and we get

$$\begin{aligned} \left.\frac{d\mathscr{P}_m}{2\pi d\theta}\right|_{\theta=\frac{\pi}{2}} &= \frac{\omega_0 e^2}{\pi R}\beta^3 m^2\left[\left(J'_m(\beta m)\right)^2 + 0\right] \\ &\overset{\beta\approx 1}{\approx} \frac{\omega_0 e^2}{\pi R}m^2\left(J'_m(m)\right)^2 \end{aligned}$$

$$\begin{aligned}\mathscr{P}_m &= \frac{e^2}{R} m\omega_0 2\beta^2 J'_{2m}(2m\beta) \\ &\overset{\beta\approx 1}{\approx} \frac{\omega_0 e^2}{R} 2mJ'_{2m}(2m)\end{aligned}$$

But $J'_m(m) \propto m^{-2/3}$ for $m \gg 1$

$$\implies \frac{d\mathscr{P}_m}{d\theta} \approx \mathscr{P}_m \cdot m \cdot m^{-2/3} = \mathscr{P}_m m^{1/3}$$

$$\implies \frac{d\mathscr{P}_m}{\mathscr{P}_m} \approx \frac{d\theta}{m^{-1/3}}$$

$\implies$ The radiation is confined to a cone about $\theta = \pi/2$ with opening angle $\Delta\theta \propto \frac{1}{m^{1/3}} \propto \frac{1}{\gamma}$.

## 6.3 Qualitative explanation of the main features

From § 6.2, synchrotron radiation is characterized by

(i) a narrow angle close to the orbital plane

(ii) high frequencies.

For a point particle with trajectory $\boldsymbol{X}(t)$, we have the Liénard?Wiechert potentials (Problem #44)

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{x},t) &= \frac{e\boldsymbol{v}(t_-)/c}{|\boldsymbol{x}-\boldsymbol{X}(t_-)| - \boldsymbol{v}(t_-)\cdot(\boldsymbol{x}-\boldsymbol{X}(t_-))/c} \\ &= \frac{e\boldsymbol{v}(t_-)/c}{|\boldsymbol{x}-\boldsymbol{X}(t_-)|}\frac{1}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{v}(t_-)/c}\end{aligned}$$

where $t_- = t - \frac{1}{c}|\boldsymbol{x}-\boldsymbol{X}(t_-)|$ and

$$\hat{\boldsymbol{n}} := \frac{\boldsymbol{x}-\boldsymbol{X}}{|\boldsymbol{x}-\boldsymbol{X}|}.$$

Let $\varphi = \sphericalangle(\boldsymbol{v},\hat{\boldsymbol{n}})$.

$$\begin{aligned}\implies \frac{1}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{v}/c} &= \frac{1}{1-\beta\cos\varphi} \\ &\overset{1.}{\to} \frac{1}{1-\beta\left(1-\frac{1}{2}\varphi^2+\dots\right)} \\ &\approx \frac{1}{\frac{1}{2}(1+\beta)(1-\beta)+\frac{1}{2}\varphi^2} \\ &= \frac{2}{1-\beta^2+\varphi^2}\end{aligned}$$

(i) Let $\beta \to 1$ and consider small $\varphi$.

$\implies$ $\boldsymbol{A}$ is appreciably nonzero for $\varphi \lesssim \sqrt{1-\beta^2} = \frac{1}{\gamma}$. This explains (i).

Consider a particle in a circular orbit. The light reaches the observer only during a section $\Delta s$ of the orbit, where

$$\frac{\Delta s}{2\pi R} \approx \frac{\varphi}{2\pi} \implies \Delta s \approx R\varphi.$$

$\implies$ The observed signal is emitted only during a time interval

$$\frac{\Delta t}{2\pi/\omega_0} \approx \frac{\Delta s}{2\pi R} \approx \frac{\varphi}{2\pi} \implies \Delta t \approx \frac{\varphi}{\omega_0}.$$

$\implies$ The typical frequency emitted is

$$\omega_e \approx \frac{1}{\Delta t} \approx \frac{\omega_0}{\varphi} \approx \omega_0\gamma. \qquad (*)$$

This explains one factor of $\gamma$.

This hold in a co-moving reference frame, but from the observer's point of view, $\Delta s$ gets Lorentz contracted by $1/\gamma \implies \omega_e$ is larger by a factor of $\gamma$.

Finally, the observer sees a Doppler shifted frequency (as discussed in Ch. 4 § 1.6) which provides the another factor of $\gamma$.

$$\implies \omega_{\text{observerd}} \approx \underbrace{\omega_0\gamma}_{(*)} \cdot \gamma \cdot \gamma = \omega_0\gamma^3$$

This explains (ii).

## 6.4 The polarization of synchrotron radiation

Polarization is measured via the effect of the $\boldsymbol{E}-$ field.

$\implies$ Express the power spectrum in terms of $\boldsymbol{E}$.

From § 4.3,

$$\frac{dU}{d\Omega} = \frac{c}{4\pi} r^2 \int \frac{d\omega}{2\pi}\, \hat{\boldsymbol{x}} \cdot \left[\boldsymbol{E}(\boldsymbol{x},\omega) \times \boldsymbol{B}(\boldsymbol{x},-\omega)\right].$$

From § 4.2 Proposition 1,

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x},\omega) &\propto \hat{\boldsymbol{x}} \times \boldsymbol{j}(\boldsymbol{k},\omega), \\ \boldsymbol{E}(\boldsymbol{x},\omega) &\approx -\hat{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x},\omega), \\ \implies \boldsymbol{E}(\boldsymbol{x},\omega) &\propto -\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{j}(\boldsymbol{k},\omega)). \end{aligned}$$

$\implies$ our previous expressions remain valid if we substitute

$$\hat{\boldsymbol{x}} \times \boldsymbol{j} \to -\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{j})$$

Now, from § 5.1,

$$\underline{\frac{d^2\mathscr{P}}{d\omega\, d\Omega}(T) = \frac{\omega^2}{4\pi^2c^3} \int d\tau\, e^{i\omega\tau} \left[-\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{j}(\boldsymbol{k}, T+\tau/2))\right] \cdot \left[-\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{j}(\boldsymbol{k}, T-\tau/2))^*\right]}$$

**Definition 1.** Set our coordinate system as in § 6.2: orbit in $x-y$ plane, $\hat{\boldsymbol{x}} = (\sin\theta, 0, \cos\theta)$.

Define *parallel polarization* as $\boldsymbol{E} \parallel \hat{\boldsymbol{e}}_{\parallel}$ where $\hat{\boldsymbol{e}}_{\parallel} = (0,1,0)$.

Define *perpendicular polarization* as $\boldsymbol{E} \parallel \hat{\boldsymbol{e}}_{\perp}$ where $\hat{\boldsymbol{e}}_{\perp} = (-\cos\theta, 0, \sin\theta)$.

We can express the radiated power in terms of these polarizations.

Power radiated into parallel polarization state:

$$\begin{aligned} \underline{\left(\frac{d^2\mathscr{P}(T)}{d\omega\, d\Omega}\right)_{\parallel}} &= \frac{\omega^2}{4\pi^2c^3} \int d\tau\, e^{i\omega\tau} [\cdots]_y [\cdots]_y \\ &\overset{1.}{=} \frac{\omega^2}{4\pi^2c^3} \int d\tau\, e^{i\omega\tau} j_y(\boldsymbol{k}, T+\tau/2)\, j_y(\boldsymbol{k}, T-\tau/2) \\ &\overset{2.}{=} \underline{\frac{\omega^2e^2}{4\pi^2c^3} \int d\tau\, e^{i\omega\tau} e^{ik\hat{\boldsymbol{x}}\cdot[\boldsymbol{y}(T+\tau/2)-\boldsymbol{y}(T-\tau/2)]} v_y(T+\tau/2)\, v_y(T-\tau/2)} \end{aligned}$$

(i) $\left[-\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{j})\right]_y = \left[\boldsymbol{j} - \hat{\boldsymbol{x}}(\hat{\boldsymbol{x}} \cdot \boldsymbol{j})\right]_y = j_y$ since $\hat{\boldsymbol{x}}$ has no $y-$component.

(ii) $\boldsymbol{j}(\boldsymbol{k},t) = e\boldsymbol{v}(t)\, e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}(t)}$.

But in § 6.2, the power had the factor $\left[\boldsymbol{v}^2 - c^2\right]$ where here we have $v_yv_y$.

**Lemma 1.**

$$\boxed{v_y(T+\tau/2)\, v_y(T-\tau/2) = \frac{1}{2}v^2\left[\cos 2\omega_0 T + \cos\omega_0\tau\right]}$$

**Lemma 2.**

$$\boxed{\overline{e^{ik\hat{\boldsymbol{x}}\cdot[\boldsymbol{y}(T+\tau/2)-\boldsymbol{y}(T-\tau/2)]}} = \sum_{m=-\infty}^{\infty} \left(J_m\left(kR\sin\theta\right)\right)^2 e^{im\omega_0\tau}},$$

*with* $\overline{f(T)}$ *averaged over one* $T-$*period.*

**Lemma 3.**

$$\boxed{\overline{\cos\left(2\omega_0 T\right) e^{ik\hat{\boldsymbol{x}}\cdot[\boldsymbol{y}(T+\tau/2)-\boldsymbol{y}(T-\tau/2)]}} = -\sum_{m=-\infty}^{\infty} J_{m+1}\left(kR\sin\theta\right) J_{m-1}\left(kR\sin\theta\right) e^{im\omega_0\tau}}$$

Substituting these into expression for power radiated into parallel polarization yields

$$\begin{aligned}
\overline{\left(\frac{d^2\mathscr{P}(T)}{d\omega\, d\Omega}\right)_{\|}} &= \frac{\omega^2 e^2}{4\pi^2 c^3}\int d\tau\, e^{i\omega\tau}\overline{e^{ik\hat{\boldsymbol{x}}\cdot[\boldsymbol{y}(T+\tau/2)-\boldsymbol{y}(T-\tau/2)]} v_y\left(T+\tau/2\right) v_y\left(T-\tau/2\right)} \\
&\overset{1.}{=} \frac{\omega^2 e^2}{4\pi^2 c^3}\int d\tau\, e^{i\omega\tau}\frac{1}{2}v^2\overline{e^{ik\hat{\boldsymbol{x}}\cdot[\boldsymbol{y}(T+\tau/2)-\boldsymbol{y}(T-\tau/2)]}\left[\cos 2\omega_0 T+\cos\omega_0\tau\right]} \\
&\overset{2.}{=} \frac{\omega^2 e^2}{4\pi^2 c^3}\frac{v^2}{2}\int d\tau\, e^{i\omega\tau}\sum_{m=-\infty}^{\infty}\left[\cos\omega_0\tau J_m^2 - J_{m+1}J_{m-1}\right] e^{im\omega_0\tau} \\
&\overset{3.}{=} \frac{\omega^2 e^2}{4\pi^2 c^3}\frac{v^2}{2}\sum_{m=-\infty}^{\infty}\left[\frac{1}{2}J_{m+1}^2 + \frac{1}{2}J_{m-1}^2 - J_{m+1}J_{m-1}\right]\delta\left(\omega - m\omega_0\right)
\end{aligned}$$

(i) Inserted Lemma 1.

(ii) Inserted Lemma 2, Lemma 3. Arguments of the Bessel functions are $kR\sin\theta$.

(iii) Replaced $\cos\omega_0\tau$ using Euler's formula.

$\Longrightarrow$ $\left(\frac{d\mathscr{P}_m}{d\Omega}\right)_{\|}$ is given by the expression for $\frac{d\mathscr{P}_m}{d\Omega}$ in § 6.2 with

$$\begin{aligned}
\frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - \frac{c^2}{v^2}J_m^2 \quad \rightarrow \quad & \frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - J_{m+1}J_{m-1} \\
= \quad & \frac{1}{2}\left(J_{m+1} - J_{m-1}\right)^2 \\
\overset{1.}{=} \quad & \frac{1}{2}4\left(J_m'\right)^2 \\
= \quad & 2\left(J_m'\right)^2
\end{aligned}$$

(i) Recursion relation.

**Theorem 1.** *The power radiated into the* $m^{\text{th}}$ *harmonic with parallel polarization is*

$$\boxed{\left(\frac{d\mathscr{P}_m}{d\Omega}\right)_{\|} = \frac{\omega_0 e^2}{\pi R}\left(\frac{v}{c}\right)^3 m^2\left(J_m'\left(m\frac{v}{c}\sin\theta\right)\right)^2}.$$

*This is the first of the two terms in* $\frac{d\mathscr{P}_m}{d\Omega}$ *from § 6.2.*

**Corollary 1.** *The power radiated into the* $m^{\text{th}}$ *harmonic with perpendicular polarization is*

$$\boxed{\left(\frac{d\mathscr{P}_m}{d\Omega}\right)_\perp = \frac{\omega_0 e^2}{\pi R}\left(\frac{v}{c}\right)^3 m^2 \left(\frac{J_m\left(m\frac{v}{c}\sin\theta\right)}{\frac{v}{c}\tan\theta}\right)^2}.$$

*This is the second of the two terms in* $\frac{d\mathscr{P}_m}{d\Omega}$ *from* § 6.2.

# Appendix A

# Glossary of notation

| Scalars | | |
|---|---|---|
| $\boldsymbol{x}\cdot\boldsymbol{y} := g(\boldsymbol{x},\boldsymbol{y})$ | $=$ | $x_j y^j$ |

| Vectors | | |
|---|---|---|
| $(\boldsymbol{x}\times\boldsymbol{y})_j$ | $:=$ | $\varepsilon_{jkl}x^k y^l$ |

| Tensors | | |
|---|---|---|
| $(\mathbb{1}_n)^j{}_k$ | $:=$ | $\delta^j_k$ |
| $(\boldsymbol{x}\otimes\boldsymbol{y})^{jk}$ | $:=$ | $x^j y^k$ |

| Matrices | | |
|---|---|---|
| $(A^T)^j{}_k$ | $:=$ | $A_k{}^j$ |

| S. fields | | | transforms as... |
|---|---|---|---|
| $(\nabla\cdot\boldsymbol{v})(\boldsymbol{x})$ | $:=$ | $\partial_j v^j(\boldsymbol{x})$ | *scalar* |

| V. fields | | | transforms as... |
|---|---|---|---|
| $(\nabla f)_j(\boldsymbol{x})$ | $:=$ | $\frac{\partial}{\partial x^j}f(\boldsymbol{x}) =: \delta_j f(\boldsymbol{x})$ | *covector* |
| $(\nabla\times\boldsymbol{v})^j(\boldsymbol{x})$ | $:=$ | $\varepsilon^{jkl}\partial_k v_l(\boldsymbol{x})$ | *pseudovector* |

# Appendix B

# Transformation identities

Let $D$ be a coordinate transformation. By Claim 1,

$$\boxed{D^j{}_k = \frac{\partial \tilde{x}^j}{\partial x^k}, \quad \left(D^{-1}\right){}^j{}_k = \frac{\partial x^j}{\partial \tilde{x}^k}}.$$

In what follows, transformation identities have been tabulated for various mathematical objects.

## 1 Scalar fields

| $CS$ | $\widetilde{CS}$ |
|---|---|
| $(\nabla \cdot \boldsymbol{v})(\boldsymbol{x}) = (\widetilde{\nabla \cdot \boldsymbol{v}})(\tilde{\boldsymbol{x}})$ | " |

## 2 Vectors

| $CS$ | $\widetilde{CS}$ |
|---|---|
| $\boldsymbol{e}_j = D^k{}_j \tilde{\boldsymbol{e}}_k$ | $\tilde{\boldsymbol{e}}_j = (D^{-1})^k{}_j \boldsymbol{e}_k$ |
| $x^j = (D^{-1})^j{}_k \tilde{x}^k$ | $\tilde{x}^j = D^j{}_k x^k$ |
| $x_j = D^k{}_j \tilde{x}_k$ | $\tilde{x}_j = \left(D^{-1}\right){}^k{}_j x_k$ |

## 3 Vector fields

| $CS$ | $\widetilde{CS}$ |
|---|---|
| $\partial_j f(\boldsymbol{x}) = D^k{}_j \tilde{\partial}_k \tilde{f}(\tilde{\boldsymbol{x}})$ | $\tilde{\partial}_j \tilde{f}(\tilde{\boldsymbol{x}}) = \left(D^{-1}\right){}^k{}_j \partial_k f(\boldsymbol{x})$ |
| $(\nabla \times \boldsymbol{v})^j(\boldsymbol{x}) = (\det D)\left(D^{-1}\right){}^j{}_k (\widetilde{\nabla \times \boldsymbol{v}})^k(\tilde{\boldsymbol{x}})$ | $(\widetilde{\nabla \times \boldsymbol{v}})^j(\tilde{\boldsymbol{x}}) = (\det D)\, D^j{}_k (\nabla \times \boldsymbol{v})^k(\boldsymbol{x})$ |

## 4 Tensors

| $CS$ | $\widetilde{CS}$ |
|---|---|
| $g_{jk} = D^m{}_j \tilde{g}_{ml} D^l{}_k$ | $\tilde{g}_{jk} = (D^{-1})^m{}_j g_{ml} (D^{-1})^l{}_k$ |
| $\varepsilon^{jkl} = (\det D)\left(D^{-1}\right){}^j{}_\alpha \left(D^{-1}\right){}^k{}_\beta \left(D^{-1}\right){}^l{}_\gamma \varepsilon^{\alpha\beta\gamma}$ | $\tilde{\varepsilon}^{jkl} = (\det D)\, D^j{}_\alpha D^k{}_\beta D^l{}_\gamma \varepsilon^{\alpha\beta\gamma}$ |

# Appendix C

# Electromagnetic field tensor

In what follows, we define

$$\boldsymbol{E} := \left(E^1, E^2, E^3\right) =: (E_x, E_y, E_z) \quad \text{and} \quad B^{jk} := \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}.$$

Note the (confusing) convention that upper numerical indices correspond to lower "Cartesian" indices. Also note that $B^{jk} = B_{jk}$.

## 1 Covariant components $F_{\mu\nu}$

$$F_{\mu\nu} =: \left(\begin{array}{c|c} 0 & \boldsymbol{E} \\ \hline -\boldsymbol{E} & B_{jk} \end{array}\right)$$

## 2 Contravariant components $F^{\mu\nu}$

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{\mu} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{\nu} F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -\boldsymbol{E} \\ \hline \boldsymbol{E} & B_{jk} \end{array}\right)$$

## 3 Mixed components ${F^{\mu}}_{\nu}$

$${F^{\mu}}_{\nu} = g^{\mu\alpha} F_{\alpha\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right) \left(\begin{array}{c|c} 0 & \boldsymbol{E} \\ \hline -\boldsymbol{E} & B_{jk} \end{array}\right) = \left(\begin{array}{c|c} 0 & \boldsymbol{E} \\ \hline \boldsymbol{E} & B_{jk} \end{array}\right)$$

## 4 Mixed components ${F_{\mu}}^{\nu}$

$${F_{\mu}}^{\nu} = g_{\mu\alpha} F^{\alpha\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1}_3 \end{array}\right) \left(\begin{array}{c|c} 0 & -\boldsymbol{E} \\ \hline \boldsymbol{E} & B^{jk} \end{array}\right) = \left(\begin{array}{c|c} 0 & -\boldsymbol{E} \\ \hline -\boldsymbol{E} & -B_{jk} \end{array}\right)$$