PHYS 623 Classical Electrodynamics S 2021

Problem Assignment # 11 04/08/2021
due 04/15/2021

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3.1.1. **Electromagnetic waves and gauge invariance**

a) Show that the Lorenz gauge, $\frac{1}{c}\,\partial_t\,\varphi + \nabla\cdot\boldsymbol{A} = 0$, still does not uniquely determine the potentials of an electromagnetic wave: Let $f$ be an arbitrary scalar solution of the wave equation, $\Box f = 0$. Then the transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla} f$, $\varphi \to \varphi - \frac{1}{c}\,\partial_t f$ leaves both the wave equation for the 4-vector potential and the fields unchanged.

b) Show in particular that the gauge of an electromagnetic wave can always be chosen such that $\varphi = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$.

(3 points)

3.1.2. **Plane waves**

Consider the scalar field

$$\psi(\boldsymbol{x}, t) = \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\ ,$$

where $\boldsymbol{k}$ is a Euclidian vector.

a) What is necessary and sufficient to make $\psi$ a solution of the wave equation?

b) Perform a Lorentz boost, and show that the transformed wave again has the form

$$\psi'(\boldsymbol{x}', t') = \cos(\boldsymbol{k}'\cdot\boldsymbol{x}' - \omega' t')\ .$$

How are $\boldsymbol{k}'$ and $\omega'$ related to $\boldsymbol{k}$ and $\omega$?

(3 points)

3.1.3. **Spherical waves**

Consider the wave equation

$$\left(\frac{1}{c^2}\,\partial_t^2 - \boldsymbol{\nabla}^2\right)\, f(\boldsymbol{x}, t) = 0$$

Find and discuss the most general solution that has the form

$$f(\boldsymbol{x}, t) = u(r, t)/r$$

where $r = |\boldsymbol{x}|$.

(3 points)

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3.1.4. **Cosmological redshift**

Edwin Hubble observed the following relation between the wavelength of spectral lines in galaxies and the distance of the galaxies from the earth:

$$(\lambda - \lambda_0)/\lambda = Hr/c$$

where $\lambda$ is the wavelength of a spectral line as observed in the galaxy, $\lambda_0$ is the wavelength of the same spectral line as measure in the laboratory, $r$ is the distance of the galaxy, and $c$ is the speed of light. $H$ is observed to be roughly $H \approx 68$ (km/s)/Mpc (1 Mpc $= 3.26 \times 10^6$ light years).

a) Assuming that the observed red shift is due to the nonrelativistic Doppler effect, and that the motion of the galay is purely radial, find a relation between the distance of a galaxy and its velocity with respect to the earth.

b) How long did it take a galaxy that's now at distance $r$ to get there? Use the result to estimate the age of the universe.

c) Hubble's original estimate was $H \approx 530$ (km/s)/Mpc. Why does this value pose a problem?

(3 points)

3.2.1. **General solution of the wave equation**

Consider a one-dimensional wave equation

$$(\partial_t^2 - c^2 \partial_x^2) f(x,t) = 0$$

Show that the general solution constructed by Fourier transform in ch. 3 §2.2 has the form of the d'Alembert solution from ch. 3 §1.2, and vice versa.

(2 points)

4.1.1. **Wave equations for the electromagnetic fields**

Show directly from the Maxwell equations, without introducing potentials, that the fields obey the inhomogeneous wave equations

$$\Box \boldsymbol{E} = -4\pi \left( \boldsymbol{\nabla} \rho + \frac{1}{c^2} \partial_t \boldsymbol{j} \right) \quad , \quad \Box \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{\nabla} \times \boldsymbol{j} \ .$$

(2 points)