Problem Assignment # 12 04/15/2021
due 04/22/2021

4.1.2. **Polaritons**

This problem doubles as the take-home midterm. Submit your solution to dbelitzuoregon.edu no later than 12 noon on Thursday, April 29. Please see my email with subject line '623 Midterm' for additional instructions.

As a model for a dielectric, consider a polarization field $\boldsymbol{P}(\boldsymbol{x}, t)$ that determines the sources of the electromagnetic fields according to

$$\boldsymbol{j} = \partial_t \boldsymbol{P} \quad , \quad \rho = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \quad .$$

In addition to Maxwell's equations, the dynamics of the system are governed by an equation of motion for $\boldsymbol{P}$,

$$\left(\partial_t^2 + \omega_0^2\right) \boldsymbol{P}(\boldsymbol{x}, t) = a^2\, \boldsymbol{E}(\boldsymbol{x}, t) \quad (*) \quad ,$$

where ω_0 is a characteristic frequency and a is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

a) Show that Maxwell's equations plus $(*)$ have solutions given by both longitudinal ($\boldsymbol{k} \parallel \boldsymbol{E}, \boldsymbol{P}$) and transverse ($\boldsymbol{k} \perp \boldsymbol{E}, \boldsymbol{P}$) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.

b) Show that the transverse waves in the long-wavelength limit are photon-like, viz., $\omega_T(\boldsymbol{k} \to 0) = (c/n)|\boldsymbol{k}|$, and determine the index of refraction n.

c) Show that no homogeneous wave propagation is possible in a frequency band $\omega_- < \omega < \omega_+$, and find $\omega_\mp$. Derive the Lyddane-Sachs-Teller relation

$$\omega_+^2/\omega_-^2 = \epsilon(\omega = 0)$$

where $\epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2)$ is the dielectric function of the dielectric.

d) Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits $k \to 0$ and $k \to \infty$, and plot the result.

(14 points)

4.2.1. **Liénard-Wiechert potentials**

Consider a point charge e that moves on a given trajectory $\boldsymbol{X}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{X}}(t)$ which results in charge and current densities

$$\rho(\boldsymbol{x}, t) = e\,\delta(\boldsymbol{x} - \boldsymbol{X}(t)) \qquad , \qquad \boldsymbol{j}(\boldsymbol{x}, t) = e\,\boldsymbol{v}(t)\,\delta(\boldsymbol{x} - \boldsymbol{X}(t))$$

Show that the resulting retarded potentials have the form

$$\varphi(\boldsymbol{x}, t) = \frac{e}{|\boldsymbol{x} - \boldsymbol{X}(t_-)| - \boldsymbol{v}(t_-) \cdot (\boldsymbol{x} - \boldsymbol{X}(t_-))/c}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\,\boldsymbol{v}(t_-)\,\varphi(\boldsymbol{x}, t)$$

where t_- is the solution of

$$t_- = t - \frac{1}{c}\,|\boldsymbol{x} - \boldsymbol{X}(t_-)| \qquad (*)$$

These are known as Liénard-Wiechert potentials after Alfred-Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

hint: Show that the equation $(*)$ for t_- has one and only one solution.

(6 points)

4.2.2. **Potential of a uniformly moving charge**

Consider a charge e moving uniformly along the x-axis with velocity v: $\boldsymbol{X}(t) = (vt, 0, 0)$. Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 2 §2.4 by means of a Lorentz transformation.

(6 points)