PHYS 622 Classical Electrodynamics W 2021

Problem Assignment # 2 01/15/2021
due 01/22/2021

0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)]\right)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

i) $F = \int dx\, \varphi(x)$

ii) $F = \int dx\, \varphi^2(x)$

iii) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
hint: Integrate by parts and assume that the boundary terms vanish.

iv) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

0.2.5. **Massive scalar field**

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi(x))\,(\partial^\mu\phi^*(x)) - \frac{m^2}{2}\,|\phi(x)|^2$$

with ϕ^* the complex conjugate of ϕ. What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(2 points)

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0.2.6. **Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields**

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

   hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\, \boldsymbol{E} \times \boldsymbol{B} / \boldsymbol{B}^2$$

   Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.2.7. **Harmonic oscillator coupled to a magnetic field**

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0, charge e). Put the oscillator in a homogeneous time-independent magnetic field $\boldsymbol{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $\boldsymbol{B}$.

(4 points)