

2.2.3. Electrostatics in d dimensions (continued)

This is a continuation of Problem #2.2.3.

- b) Calculate and plot the potential φ and the field \mathbf{E} for $d = 2$ for the case of a homogeneously charged disk, $\rho(\mathbf{x}) = \rho_0 \Theta(r_0 - |\mathbf{x}|)$.

hint: It is easiest to proceed as in the 3- d case, see Problem 2.2.2.

note: This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

- c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0 \Theta(x_0^2/4 - x^2)$.

hint: Integrate Poisson's formula directly. (8 points)

2.2.4. Helmholtz equation

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \nabla^2)\varphi(\mathbf{x}) = 4\pi\rho(\mathbf{x})$$

in terms of an integral.

hint: The answer is a generalization of Poisson's formula.

(3 points)

2.3.1. Quadrupole moments (to be continued later)

- a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\mathbf{x}) = \frac{1}{r} Q + \frac{1}{r^3} \mathbf{x} \cdot \mathbf{d} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \dots$$

with Q the total charge and \mathbf{d} the dipole moment, and determine the quadrupole tensor Q_{ij} .

- b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

- c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

- d) Let the charge density be invariant under rotations about the z -axis through multiples of an angle α ,

with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your

result from part c) conforms with this for the special case $a = b$.

(7 points)