Problem Assignment # 9 03/05/2021
due 03/12/2021

2.3.2. **Legendre polynomials**

Consider the ODE

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

with λ a constant. Show that a necessary condition for the existence of a polynomial solution is

$$\lambda = n(n+1)$$

with $n = 0, 1, \ldots$ What else do you need to require in order to get a condition that is necessary and sufficient? Convince yourself that these considerations correctly produce the first three Legendra polynomials up to an overall normalization factor.

hint: Make a power-series ansatz and require that the series terminates.

(4 points)

2.3.3. **Associated Legendre functions**

note: When comparing with the reference book by Abramowitz and Stegun, note that their $P_\ell^m(x)$ equals $(-)^{3m/2}$ times our $P_\ell^m(x)$.

Show that

$$\left(\sqrt{1-x^2}\,\frac{d}{dx} - m\,\frac{x}{\sqrt{1-x^2}}\right) P_\ell^m(x) = (\ell+m)(\ell-m+1)\,P_\ell^{m-1}(x)$$

hint: First differentiate Legendre's ODE $m-1$ times to show that

$$(1-x^2)\,\frac{d^{m+1}}{dx^{m+1}}\,P_n(x) - 2mx\,\frac{d^m}{dx^m}\,P_n(x) + (n+m)(n-m+1)\,\frac{d^{m-1}}{dx^{m-1}}\,P_n(x) = 0$$

Then use this in evaluating $\sqrt{1-x^2}\,dP_\ell^m(x)/dx$.

(3 points)

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2.3.4. **Spherical harmonics**

Prove that the sperical harmonics have the following properties:

$$
\begin{aligned}
Y_\ell^m(\Omega)^* &= (-)^m Y_\ell^{-m}(\Omega) & (1)\\
\cos\theta\, Y_\ell^m(\Omega) &= \left(\frac{(\ell+1-m)(\ell+1+m)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^m(\Omega) + \left(\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^m(\Omega) & (2)\\
\sin\theta\, e^{\pm i\varphi}\, Y_\ell^m(\Omega) &= \pm\left(\frac{(\ell\mp m-1)(\ell\mp m)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^{m\pm1}(\Omega) \mp \left(\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^{m\pm1}(\Omega) & (3)\\
\hat{L}_\mp\, Y_\ell^m(\Omega) &= ((\ell\pm m)(\ell\mp m+1))^{1/2}\, Y_\ell^{m\mp1}(\Omega) & (4)
\end{aligned}
$$

where

$$\hat{L}_\mp = e^{\mp i\varphi}\left[\mp\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right]$$

hint: Use the properties of the associated Legendre functions we quoted in ch.3 §3.2, as well as Problem 2.3.3.

(9 points)

2.3.5. **Field due to distant charges**

Consider the electric field generated by a charge density $\rho(\boldsymbol{y})$ that vanishes inside a sphere with radius r_0: $\rho(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \le r_0$. Show that

a) If ρ is invariant under parity operations, $\rho(-\boldsymbol{y}) = \rho(\boldsymbol{y})$, then the electric field at the origin vanishes.

b) If $\rho(\boldsymbol{y})$ is invariant under rotations about the z-axis through multiples of an angle α with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form $\varphi_{ij}(\boldsymbol{x}=0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$

c) If $\rho(\boldsymbol{y})$ has cubic symmetry, i.e., if $\rho(\boldsymbol{y})$ is invariant under rotations through $\pi/2$ about any of the three axes x, y, and z, then the field-gradient tensor at the origin vanishes.

(6 points)