PHYS 623 Classical Electrodynamics S 2021

Problem Assignment # 11 04/08/2021
due 04/15/2021

3.1.1. **Electromagnetic waves and gauge invariance**

a) Show that the Lorenz gauge, $\frac{1}{c}\,\partial_t\,\varphi + \nabla\cdot\boldsymbol{A} = 0$, still does not uniquely determine the potentials of an electromagnetic wave: Let f be an arbitrary scalar solution of the wave equation, $\Box f = 0$. Then the transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla} f$, $\varphi \to \varphi - \frac{1}{c}\,\partial_t f$ leaves both the wave equation for the 4-vector potential and the fields unchanged.

b) Show in particular that the gauge of an electromagnetic wave can always be chosen such that $\varphi = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$.

(3 points)

3.1.2. **Plane waves**

Consider the scalar field

$$\psi(\boldsymbol{x}, t) = \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\ ,$$

where $\boldsymbol{k}$ is a Euclidian vector.

a) What is necessary and sufficient to make ψ a solution of the wave equation?

b) Perform a Lorentz boost, and show that the transformed wave again has the form

$$\psi'(\boldsymbol{x}', t') = \cos(\boldsymbol{k}'\cdot\boldsymbol{x}' - \omega' t')\ .$$

How are $\boldsymbol{k}'$ and ω' related to $\boldsymbol{k}$ and ω?

(3 points)

3.1.3. **Spherical waves**

Consider the wave equation

$$\left(\frac{1}{c^2}\,\partial_t^2 - \boldsymbol{\nabla}^2\right)\, f(\boldsymbol{x}, t) = 0$$

Find and discuss the most general solution that has the form

$$f(\boldsymbol{x}, t) = u(r, t)/r$$

where $r = |\boldsymbol{x}|$.

(3 points)

. . . **/over**

3.1.4. **Cosmological redshift**

Edwin Hubble observed the following relation between the wavelength of spectral lines in galaxies and the distance of the galaxies from the earth:

$$(\lambda - \lambda_0)/\lambda = Hr/c$$

where λ is the wavelength of a spectral line as observed in the galaxy, λ_0 is the wavelength of the same spectral line as measure in the laboratory, r is the distance of the galaxy, and c is the speed of light. H is observed to be roughly $H \approx 68$ (km/s)/Mpc (1 Mpc $= 3.26 \times 10^6$ light years).

a) Assuming that the observed red shift is due to the nonrelativistic Doppler effect, and that the motion of the galay is purely radial, find a relation between the distance of a galaxy and its velocity with respect to the earth.

b) How long did it take a galaxy that's now at distance r to get there? Use the result to estimate the age of the universe.

c) Hubble's original estimate was $H \approx 530$ (km/s)/Mpc. Why does this value pose a problem?

(3 points)

3.2.1. **General solution of the wave equation**

Consider a one-dimensional wave equation

$$(\partial_t^2 - c^2 \partial_x^2) f(x,t) = 0$$

Show that the general solution constructed by Fourier transform in ch. 3 §2.2 has the form of the d'Alembert solution from ch. 3 §1.2, and vice versa.

(2 points)

4.1.1. **Wave equations for the electromagnetic fields**

Show directly from the Maxwell equations, without introducing potentials, that the fields obey the inhomogeneous wave equations

$$\Box \boldsymbol{E} = -4\pi \left(\boldsymbol{\nabla} \rho + \frac{1}{c^2} \partial_t \boldsymbol{j} \right) \qquad , \qquad \Box \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{\nabla} \times \boldsymbol{j} \ .$$

(2 points)

3.1.1.) a) Let $\partial_\mu \partial^\mu f = 0$ and consider $\vec{A}' = \vec{A} + \vec{\nabla} f$

$\varphi' = \varphi - \frac{1}{c} \partial_t f$

$\rightsquigarrow \partial_\mu \partial^\mu \vec{A}' = \partial_\mu \partial^\mu \vec{A} - \frac{1}{c} \partial_t \underbrace{\partial_\mu \partial^\mu f}_{=0} + \vec{\nabla} \underbrace{\partial_\mu \partial^\mu f}_{=0} = \partial_\mu \partial^\mu \vec{A} = 0$

1 $\rightsquigarrow$ $\vec{A}'$ also obeys the wave eq.

Furthermore,

$\vec{E}' = -\vec{\nabla}\varphi' - \frac{1}{c}\partial_t \vec{A}' = -\vec{\nabla}\varphi - \frac{1}{c}\partial_t \vec{A} + \cancel{\frac{1}{c}\vec{\nabla}\partial_t f} - \cancel{\frac{1}{c}\partial_t \vec{\nabla} f} = \vec{E}$

$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{B} + \vec{\nabla} \times \vec{\nabla} f = \vec{B}$ since $\vec{\nabla} \times \vec{\nabla} f = 0$

1 $\rightsquigarrow$ The fields are unchanged.

b) Choose f to be a solution of $\frac{1}{c}\partial_t f = \varphi$ $\rightsquigarrow$ $\underline{\underline{\varphi' = 0}}$

$\rightsquigarrow \underline{\underline{\vec{\nabla} \cdot \vec{A}'}} = \vec{\nabla} \cdot \vec{A} + \vec{\nabla}^2 f \overset{\Box f = 0}{=} \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\partial_t^2 f = \vec{\nabla} \cdot \vec{A} + \frac{1}{c}\partial_t \varphi \; \underline{\underline{= 0}}$

1

Lorenz gauge

3.1.2) a) $\psi(\vec{x},t) = \cos(\vec{k}\cdot\vec{x} - \omega t)$

$$\frac{1}{c^2}\partial_t^2 \psi = \frac{1}{c^2}\omega^2 \psi$$

$$\vec{\nabla}^2 \psi = \vec{k}^2 \psi$$

$$\rightarrow \quad 0 = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\psi = \left(\frac{\omega^2}{c^2} - \vec{k}^2\right)\psi$$

$$\Leftrightarrow \quad \boxed{\omega^2 = c^2\vec{k}^2}$$ necessary and sufficient for ψ to solve the wave eq.

(1)

b) Lorentz boost along x-axis:

$x' = \gamma(x - vt)$ $\quad y' = y \quad z' = z$ $\quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

$t' = \gamma(t - \frac{v}{c^2}x)$

$$\rightarrow \quad \vec{k}\cdot\vec{x} = k_x\gamma(x' + vt') + k_y y' + k_z z'$$

$$\omega t = \omega\gamma(t' + \frac{v}{c^2}x')$$

$$\rightarrow \quad \underline{\vec{k}\cdot\vec{x} - \omega t} = \left(k_x\gamma - \omega\gamma\frac{v}{c^2}\right)x' + k_y y' + k_z z' - \omega\gamma t' + k_x\gamma v t'$$

$$= \gamma\left(k_x - \omega\frac{v}{c^2}\right)x' + k_y y' + k_z z' - \gamma(\omega - k_x v)t'$$

(1)

$$\rightarrow \quad \underline{\underline{\psi'(\vec{x}',t')}} = \underline{\underline{\cos(\vec{k}'\cdot\vec{x}' - \omega' t')}}$$

where $\omega'/c = \gamma(\omega/c - \frac{v}{c}k_x)$

$k_x' = \gamma(k_x - \frac{v}{c}\omega/c)$ $\quad k_y' = k_y \quad k_z' = k_z$

$\rightarrow$ $\underline{(\omega/c, \vec{k})}$ transforms as a $\underline{\text{Minkowski vector}}$

(1)

3.1.3.) wave eq.:

$$0 = \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right) f(\vec{x},t) =$$

$$= \left(\frac{1}{c^2}\partial_t^2 - \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\Delta\right) f(r,\vartheta,\varphi;t)$$

in spherical coordinates, with Δ from ch 3 §4.2 remark (8).

1

ansatz: $f(r,\Omega;t) = \frac{1}{r} u(r,\cancel{\Omega};t)$

$$\rightsquigarrow \frac{1}{r}\partial_r^2 r f = \frac{1}{r}\partial_r^2 u$$

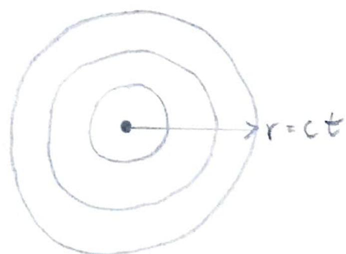
$$\rightsquigarrow \frac{1}{c^2}\partial_t^2 \frac{1}{r} u - \frac{1}{r^2}\partial_r^2 u = 0$$

$$\rightsquigarrow \boxed{\left(\frac{1}{c^2}\partial_t^2 - \partial_r^2\right) u(r,t) = 0}$$

This is the plane-wave eq. from ch 5 §1.2

1 general solution: $\underline{u(r,t) = f_1(r-ct) + f_2(r+ct)}$

with f_1, f_2 arbitrary fcts.



Then one spherical waves, i.e., the surfaces of constant phase are concentric spheres $\underline{r = \pm ct + \text{const.}}$

1

3.1.4.) $(\lambda-\lambda_0)/\lambda = Hr/c$ $\quad$ H = Hubble constant

a) $\lambda = 2\pi/k = 2\pi c/\omega$

Doppler effect: $\omega = \omega_0(1-\frac{v}{c}\cos\vartheta)$ (nonrelativistic)

radial motion: $\vartheta = 0 \rightarrow \cos\vartheta = 1$

$\rightarrow \frac{2\pi c}{\lambda} = \frac{2\pi c}{\lambda_0}(1-v/c)$

$\rightarrow \lambda_0 = \lambda - \lambda v/c$

$\rightarrow (\lambda-\lambda_0)/\lambda = v/c$

(1) $\rightarrow$ $\underline{v = Hr}$

b) $t = r/v = 1/H$ same time for all galaxies

$\rightarrow$ The expansion started a time t_0 ago, with

$\underline{t_0} = 1/H \simeq \frac{1}{68}\,\frac{s}{km}\,\text{Mpc} = \frac{1}{68}\,3.26\times10^6 \times \frac{9.46\times10^{17}}{10^5}\,s$

(1) $= 4.54\times10^{17}\,s \simeq \underline{1.4\times10^{10}\,\text{yrs}}$ estimated age of the universe

c) If H were larger by a factor of 10, then t_0 would be $\simeq 10^9$ yrs, which is shorter than the age of the earth by a factor of 5 !

(1)

3.2.1.) use §2.2 →

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left[f_\lambda^+ e^{i(\lambda x - \omega_\lambda t)} + f_\lambda^- e^{-i(\lambda x - \omega_\lambda t)} \right]$$

(1) with $\omega_\lambda = c|\lambda|$.

$$\to 2\pi f(x,t) = \int_0^\infty d\lambda \left[f_\lambda^+ e^{i(\lambda x - c\lambda t)} + f_\lambda^- e^{-i(\lambda x - c\lambda t)} \right]$$
$$+ \int_{-\infty}^0 d\lambda \left[f_\lambda^+ e^{i(\lambda x + c\lambda t)} + f_\lambda^- e^{-i(\lambda x + c\lambda t)} \right]$$
$$= \int_0^\infty d\lambda \left[f_\lambda^+ + f_\lambda^- \right] e^{i\lambda(x-ct)} + \int_0^\infty d\lambda \left[f_{-\lambda}^+ + f_{-\lambda}^- \right] e^{-i\lambda(x+ct)}$$
$$= 2\pi f_1(x-ct) + 2\pi f_2(x+ct)$$

with
$$f_1(x) = \frac{1}{2\pi}\int_0^\infty d\lambda \left[f_\lambda^+ + f_\lambda^- \right] e^{i\lambda x}$$
$$f_2(x) = \frac{1}{2\pi}\int_0^\infty d\lambda \left[f_{-\lambda}^+ + f_{-\lambda}^- \right] e^{-i\lambda x}$$

(1) this is the d'Alembert solution

4.1.1.) $\Box\vec{B} = \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right)\vec{B} =$

$= \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\vec{B}\right) + \vec{\nabla}\times\vec{\nabla}\times\vec{B}$ via $\vec{\nabla}\times\vec{\nabla}\times\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v}$ and $\vec{\nabla}\cdot\vec{B} = 0$

$= \cancel{\frac{1}{c}\partial_t\left(-\vec{\nabla}\times\vec{E}\right)} + \frac{4\pi}{c}\vec{\nabla}\times\vec{j} + \cancel{\frac{1}{c}\partial_t\vec{\nabla}\times\vec{E}}$ by M-eqs (2)+(4)

$= \frac{4\pi}{c}\vec{\nabla}\times\vec{j}$

(1)

$\frac{1}{c^2}\partial_t^2\vec{E} = \frac{1}{c}\partial_t\left(-\frac{4\pi}{c}\vec{j} + \vec{\nabla}\times\vec{B}\right)$ by M-eq. (4)

$= -\frac{4\pi}{c^2}\partial_t\vec{j} + \vec{\nabla}\times\left(-\vec{\nabla}\times\vec{E}\right)$ by M-eq. (2)

$= -\frac{4\pi}{c^2}\partial_t\vec{j} - \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \vec{\nabla}^2\vec{E}$

$= -\frac{4\pi}{c^2}\partial_t\vec{j} - 4\pi\vec{\nabla}\rho + \vec{\nabla}^2\vec{E}$ by M-eq. (1)

$\rightarrow \Box\vec{E} = -4\pi\left(\vec{\nabla}\rho + \frac{1}{c^2}\partial_t\vec{j}\right)$

(1)