Problem Assignment # 12 04/15/2021
due 04/22/2021

4.1.2. **Polaritons**

As a model for a dielectric, consider a polarization field $\boldsymbol{P}(\boldsymbol{x}, t)$ that determines the sources of the electromagnetic fields according to

$$\boldsymbol{j} = \partial_t \boldsymbol{P} \quad , \quad \rho = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \quad .$$

In addition to Maxwell's equations, the dynamics of the system are governed by an equation of motion for $\boldsymbol{P}$,

$$\left(\partial_t^2 + \omega_0^2\right) \boldsymbol{P}(\boldsymbol{x}, t) = a^2\, \boldsymbol{E}(\boldsymbol{x}, t) \quad (*) \quad ,$$

where ω_0 is a characteristic frequency and a is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

a) Show that Maxwell's equations plus $(*)$ have solutions given by both longitudinal ($\boldsymbol{k} \parallel \boldsymbol{E}, \boldsymbol{P}$) and transverse ($\boldsymbol{k} \perp \boldsymbol{E}, \boldsymbol{P}$) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.

b) Show that the transverse waves in the long-wavelength limit are photon-like, viz., $\omega_T(\boldsymbol{k} \to 0) = (c/n)|\boldsymbol{k}|$, and determine the index of refraction n.

c) Show that no homogeneous wave propagation is possible in a frequency band $\omega_- < \omega < \omega_+$, and find $\omega_\mp$. Derive the Lyddane-Sachs-Teller relation

$$\omega_+^2/\omega_-^2 = \epsilon(\omega = 0)$$

where $\epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2)$ is the dielectric function of the dielectric.

d) Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits $k \to 0$ and $k \to \infty$, and plot the result.

(14 points)

. . . **/over**

4.2.1. **Liénard-Wiechert potentials**

Consider a point charge e that moves on a given trajectory $\boldsymbol{X}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{X}}(t)$ which results in charge and current densities

$$\rho(\boldsymbol{x}, t) = e\,\delta(\boldsymbol{x} - \boldsymbol{X}(t)) \qquad , \qquad \boldsymbol{j}(\boldsymbol{x}, t) = e\,\boldsymbol{v}(t)\,\delta(\boldsymbol{x} - \boldsymbol{X}(t))$$

Show that the resulting retarded potentials have the form

$$\varphi(\boldsymbol{x}, t) = \frac{e}{|\boldsymbol{x} - \boldsymbol{X}(t_-)| - \boldsymbol{v}(t_-) \cdot (\boldsymbol{x} - \boldsymbol{X}(t_-))/c}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\,\boldsymbol{v}(t_-)\,\varphi(\boldsymbol{x}, t)$$

where t_- is the solution of

$$t_- = t - \frac{1}{c}\,|\boldsymbol{x} - \boldsymbol{X}(t_-)| \qquad (*)$$

These are known as Liénard-Wiechert potentials after Alfred-Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

hint: Show that the equation $(*)$ for t_- has one and only one solution.

(6 points)

4.2.2. **Potential of a uniformly moving charge**

Consider a charge e moving uniformly along the x-axis with velocity v: $\boldsymbol{X}(t) = (vt, 0, 0)$. Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 2 §2.4 by means of a Lorentz transformation.

(6 points)

4.2.1.) ch 5 $3 =>

$$\varphi(\vec{x},t) = \int d\vec{x}' dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t-t'-\tfrac{1}{c}|\vec{x}-\vec{x}'|\right) \rho(\vec{x}',t')$$

$$= \int d\vec{x}' dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t-t'-\tfrac{1}{c}|\vec{x}-\vec{x}'|\right) e\, \delta(\vec{x}'-\vec{X}(t'))$$

$$= e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \delta\left(t'-t+\tfrac{1}{c}|\vec{x}-\vec{X}(t')|\right)$$

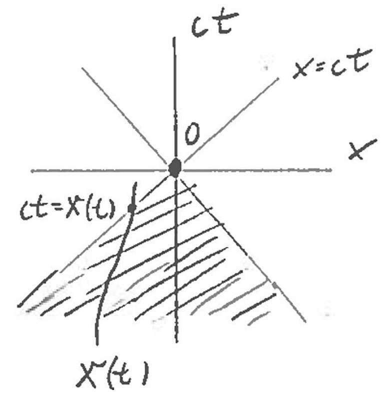
Define $f(t') = t'-t+\frac{1}{c}|\vec{x}-\vec{X}(t')|$

→ Need the zeros of $f(t')$.

let $(ct,\vec{x}) = (0,0)$ wlog.

→ $ct' = |\vec{X}(t')|$

solution = intersection between world line of particle with the light cone



speed of particle < c → there is one and only one intersection point

$$\to \delta(f(t')) = \frac{1}{|f'(t_-)|}\delta(t'-t_-)$$

when $f(t_-)=0 \Leftrightarrow \boxed{t_- = t - \frac{1}{c}|\vec{x}-\vec{X}(t_-)|}$ (*)

Discussion above → (*) has a unique solution

and

$$f'(t') = \frac{d}{dt'}f(t') = 1 + \frac{1}{c}\frac{1}{|\vec{x}-\vec{X}(t')|}(-)\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))$$

$$= 1 - \frac{1}{c}\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))\frac{1}{|\vec{x}-\vec{X}(t')|} > 0 \text{ since } \frac{|\vec{v}|}{c} < 1$$

$$\rightarrow \varphi(\vec{x},t) = e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \frac{1}{1-\frac{1}{c}\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))\frac{1}{|\vec{x}-\vec{X}(t')|}} \delta(t'-t_-)$$

$$= \frac{e}{|\vec{x}-\vec{X}(t_-)| - \frac{1}{c}\vec{v}(t_-)\cdot(\vec{x}-\vec{X}(t_-))}$$

(1)

For $\vec{A}$ we have

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d\vec{x}'\,dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta(t-t'-\tfrac{1}{c}|\vec{x}-\vec{x}'|)\,\vec{j}(\vec{x}',t')$$

$$= \frac{e}{c}\int d\vec{x}'\,dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta(f(t'))\,\vec{v}(t')\,\delta(\vec{x}'-\vec{X}(t'))$$

$$= \frac{e}{c}\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \vec{v}(t') \frac{1}{|f'(t_-)|} \delta(t'-t_-)$$

$$= \frac{1}{c}\vec{v}(t_-)\, e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \frac{1}{|f'(t_-)|} \delta(t'-t_-)$$

$$= \frac{1}{c}\vec{v}(t_-)\,\varphi(\vec{x},t)$$

(1)

=> 4.2.1

4.2.2.) Consider Problem 35 for the special case

$$\vec{x}(t) = (vt, 0, 0)\ , \quad \vec{v}(t) = (v, 0, 0)$$

-> The eq. for t_- reads

(1) $$t_- = t - \frac{1}{c}\sqrt{(x-vt_-)^2+y^2+z^2}$$

~> $$x - vt_- = x - vt - \frac{v}{c}\sqrt{(x-vt_-)^2+y^2+z^2}$$

~> $$(x-vt_-)^2 - 2(x-vt)(x-vt_-) + (x-vt)^2 = \frac{v^2}{c^2}(x-vt_-)^2 + \frac{v^2}{c^2}(y^2+z^2)$$

~> $$(x-vt_-)^2\gamma^{-2} - 2(x-vt)(x-vt_-) + (x-vt)^2 - \frac{v^2}{c^2}(y^2+z^2) = 0$$

$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

~> $$\underline{x-vt_-} = \frac{1}{2}\gamma^2\left[+2(x-vt) \pm \sqrt{4(x-vt)^2 - 4\gamma^{-2}(x-vt)^2 + 4\gamma^{-2}\frac{v^2}{c^2}(y^2+z^2)}\right]$$

$$= \gamma^2\left[x-vt \pm \sqrt{\frac{v^2}{c^2}(x-vt)^2 + \frac{v^2}{c^2}(y^2+z^2)\gamma^{-2}}\right]$$

(1) $$= \gamma^2(x-vt) \underset{(-)}{+} \frac{v}{c}\gamma^2\sqrt{(x-vt)^2+\gamma^{-2}(y^2+z^2)}$$

The physical (=retarded) solution yields the smaller value for t_- ~> The physical solution has "+" in the above eq.

Define $R^*(\vec{x},t) := \sqrt{(x-vt)^2+\gamma^{-2}(y^2+z^2)}$ as in cf. J § 2.4

(1) ~> $$\underline{x-vt_- = \gamma^2\left[x-vt+\frac{v}{c}R^*(\vec{x},t)\right]}$$ This is the explicit solution for t_-.

Problem 35 ~>

(1) $$\frac{e}{\varphi(\vec{x},t)} = \sqrt{(x-vt_-)^2+y^2+z^2} - \frac{v}{c}(x-vt_-)$$

cf. J § 2.4 ~> I need to show that the rhs equals R^*.

$\rightarrow \quad \sqrt{(x-vt_-)^2+y^2+z^2} - \frac{v}{c}(x-vt_-) \overset{?}{=} R^*$

$\rightarrow \quad (x-vt_-)^2+y^2+z^2 \overset{!}{=} \left(R^*+\frac{v}{c}(x-vt_-)\right)^2$

$$= \frac{v^2}{c^2}(x-vt_-)^2 + 2\frac{v}{c}(x-vt_-)R^* + (R^*)^2$$

$\rightarrow \quad (x-vt_-)^2\left(1-\frac{v^2}{c^2}\right) + y^2+z^2 \overset{?}{=} (R^*)^2 + 2\frac{v}{c}(x-vt_-)R^*$

$\rightarrow \quad \gamma^2\left[(x-vt)+\frac{v}{c}R^*\right]^2 + y^2+z^2 \overset{?}{=} (R^*)^2 + 2\frac{v}{c}\gamma^2\left[(x-vt)+\frac{v}{c}R^*\right]R^*$

$\rightarrow \quad \gamma^2(x-vt)^2 + \cancel{2\gamma^2\frac{v}{c}(x-vt)R^*} + \cancel{\gamma^2\frac{v^2}{c^2}(R^*)^2} + y^2+z^2 \overset{?}{=} (R^*)^2 + \cancel{2\frac{v}{c}\gamma^2(x-vt)R^*} + \cancel{2}\frac{v^2}{c^2}\gamma^2(R^*)^2$

$\rightarrow \quad \gamma^2(x-vt)^2+y^2+z^2 \overset{?}{=} (R^*)^2\left(1+\frac{v^2}{c^2}\gamma^2\right)$

$$= (R^*)^2\left(1+\frac{v^2/c^2}{1-v^2/c^2}\right) = (R^*)^2\frac{1}{1-v^2/c^2} = \gamma^2(R^*)^2$$

(1) $\rightarrow \quad \underline{(R^*)^2 \overset{!}{=} (x-vt)^2 + \gamma^{-2}(y^2+z^2)}$ ✓

Now I have

$$\frac{e}{\varphi(\vec{x},t)} = \sqrt{(x-vt_-)^2+y^2+z^2} - \frac{v}{c}(x-vt_-) = R^*(\vec{x},t)$$

$$\rightarrow \quad \underline{\underline{\varphi(\vec{x},t)}} = \underline{\underline{\frac{e}{R^*(\vec{x},t)}}}$$

and from Problem 4.2.1

$$\underline{\underline{\vec{A}(\vec{x},t)}} = \frac{\vec{v}}{c}\varphi(\vec{x},t) = \underline{\underline{\frac{e\vec{v}}{cR^*(\vec{x},t)}}}$$

(1) These are indeed the same results as in cl J § 2.6

<u>remark</u>: solution via Lorentz trafo, as in cl J § 2.6, is much easier!