Problem Assignment # 13

04/22/2021
due 04/29/2021

4.3.1. **Potentials in Coulomb gauge**

Consider the potentials φ and $\boldsymbol{A}$ in the Coulomb gauge, i.e., the field equations from ch.4 §1.2 proposition 2. Show explicitly that the resulting asymptotic electric and magnetic fields are the same as those calculated in the Lorenz gauge in ch.4 §3.

hint: Show that the scalar potential does not contribute to the electric field, and show that the asymptotic vector potential now reads

$$\boldsymbol{A}(\boldsymbol{x},t) = -\hat{\boldsymbol{x}} \times \left[\hat{\boldsymbol{x}} \times \frac{1}{rc} \int d\boldsymbol{y}\, \boldsymbol{j}(\boldsymbol{y}, t_r)\right]$$

instead of the expression derived in ch. 4 §3.1. Then calculate the fields.

(8 points)

4.3.2. **Radiation from cyclotron motion**

Consider a point mass m with charge e that moves in a plane perpendicular to a homogeneous magnetic field $\boldsymbol{B}$. Assume nonrelativistic motion, $v \ll c$

a) Find the power radiated by the particle.

b) Show that the energy of the particle decreases with time according to $E(t) = E_0\, e^{-t/\tau}$, and determine the timescale τ.

c) Find τ in seconds for an electron in a magnetic field of 1 Tesla.

(4 points)

4.3.3. **Radiating harmonic oscilator**

Consider particle with charge e and mass m in a one-dimensional harmonic potential. Let the frequency of the harmonic oscillator by ω_0.

a) Find the power radiated by the particle, averaged over one oscillation period, as a function of the energy E of the oscillator.

   hint: Remember the virial theorem, which for a harmonic potential says $\bar{V} = \bar{T} = E/2$, with V, T, and E the potential, kinetic, and total energy, respectively, of the particle, and the bar denoting a time average.

b) Show that the energy of the oscillator again decreases exponentially, $E(t) = E_0\, e^{-t/\tau}$.

c) Determine τ in seconds for e and m the electron charge and mass, respectively, and $\omega_0 = 10^{15}\,\text{sec}^{-1}$ (a typical atomic frequency).

(4 points)

4.3.1.) ch 4 $1.2 =>

In Coulomb gauge we have

$$\vec{\nabla}^2 \varphi = -4\pi \rho \qquad (*)$$

$$\Box \vec{A} = \frac{4\pi}{c} \vec{j} - \frac{1}{c} \partial_t \vec{\nabla} \varphi \qquad (**)$$

with the condition $\vec{\nabla} \cdot \vec{A} = 0$.

(*) is solved by Poisson's formula

$$\varphi(\vec{x},t) = \int d\vec{y} \, \frac{\rho(\vec{y},t)}{|\vec{x}-\vec{y}|}$$

-> Asymptotically $\varphi(\vec{x},t) = \frac{1}{r} \int d\vec{y} \, \rho(\vec{y},t) + O(1/r^2) \qquad (+)$

-> $\vec{\nabla} \varphi = O(1/r^2)$

-> The scalar potential cannot contribute to the asymptotic electric field $\vec{E}$, which decays as $1/r$.

-> For the asymptotic fields, I have

$$\vec{E}(\vec{x},t) = -\frac{1}{c} \partial_t \vec{A}(\vec{x},t) + O(1/r^2)$$
$$\vec{B}(\vec{x},t) = \vec{\nabla} \times \vec{A}(\vec{x},t)$$

(1)

Now consider (**) for $\vec{A}$. For the source $\vec{j}$ and $\vec{\nabla}\varphi$.

Do a spatial Fourier transfo on (*):

$$-\vec{k}^2 \varphi(\vec{k},t) = -4\pi \rho(\vec{k},t)$$

But charge conservation implies $\partial_t \rho(\vec{x},t) = -\vec{\nabla} \cdot \vec{j}(\vec{x},t)$

-> $\partial_t \rho(\vec{k},t) = i\vec{k} \cdot \vec{j}(\vec{k},t)$

(1) -> $\partial_t \varphi(\vec{k},t) = \frac{4\pi}{k^2} \, i\vec{k} \cdot \vec{j}(\vec{k},t)$

Use this in (**) ~>

$$\left(\frac{1}{c^2}\partial_t^2 + \vec{k}^2\right)\vec{A}(\vec{k},t) = \frac{4\pi}{c}\vec{j}(\vec{k},t) - \frac{1}{c}(-i\vec{k})\frac{4\pi}{k^2}\, i\vec{k}\cdot\vec{j}(\vec{k},t)$$

$$= \frac{4\pi}{c}\left[\vec{j}(\vec{k},t) - \hat{k}(\hat{k}\cdot\vec{j}(\vec{k},t))\right]$$

(1)

But $\vec{k} \parallel \vec{x}$ ~> $\hat{k} = \hat{x}$

~> Fourier backtransforming yields

$$\Box\vec{A}(\vec{x},t) = \frac{4\pi}{c}\left[\vec{j}(\vec{x},t) - \hat{x}(\hat{x}\cdot\vec{j}(\vec{x},t))\right]$$

$$= -\frac{4\pi}{c}\hat{x}\times(\hat{x}\times\vec{j}(\vec{x},t))$$

(1)

Solving this as in ch 5 §2 yields, in place of the expression in §3.1,

$$\boxed{\vec{A}(\vec{x},t) = -\hat{x}\times\left[\hat{x}\times\frac{1}{rc}\int d\vec{y}\;\vec{j}(\vec{y},t_r)\right]}$$

(1)

Now calculate the fields. I have

$$-\hat{x}\times(\hat{x}\times\vec{j}) = \vec{j} - \hat{x}(\hat{x}\cdot\vec{j}) = \vec{j}_T$$

where $\vec{j}_T = \vec{j} - \hat{x}(\hat{x}\cdot\vec{j})$ is the transverse part

$\equiv \vec{j} - \vec{j}_L$ with $\vec{j}_L = \hat{x}(\hat{x}\cdot\vec{j})$ the longitudinal part

But the curl is purely transverse

~> $\vec{\nabla}\times\vec{j}_L = 0$

~> $\vec{\nabla}\times\vec{j}_T = \vec{\nabla}\times\vec{j}$

$$\rightsquigarrow \quad \underline{\underline{\vec{B}(\vec{x},t)}} = \vec{\nabla}\times\vec{A}(\vec{x},t) = \vec{\nabla}\times\frac{1}{rc}\int d\vec{y}\,\vec{j}(\vec{y},t_r) + O(1/r^2)$$

1 which is the same result as in ch 5 § 3.1.

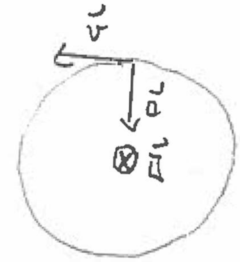
For the electric field, we have

$$\underline{\underline{\vec{E}(\vec{x},t)}} = -\frac{1}{c}\partial_t\vec{A}(\vec{x},t) = \frac{1}{c^2 r}\,\hat{x}\times\left[\hat{x}\times\int d\vec{y}\,\vec{j}(\vec{y},t_r)\right]$$

which is again the same result as in ch 5 § 3.1, and hence

1 $$\underline{\vec{E}(\vec{x},t) = -\hat{x}\times\vec{B}(\vec{x},t)}$$

remark: $\vec{A}$ in Coulomb gauge is the transverse part of $\vec{A}$ in Lorenz gauge, and the contribution $-\vec{\nabla}\varphi$ to $\vec{E}$ in Lorenz gauge makes up for the difference: It cancels the longitudinal part of $-\frac{1}{c}\partial_t\vec{A}$ in Lorenz gauge, see the proof of the proposition in ch 5 § 3.1



4.3.2.a) Lorentz force: $\vec{F}_L = \frac{e}{c}\vec{v}\times\vec{B} = -\frac{eB}{c} v\hat{r}$

Newton's 2nd law: $m\vec{a} = m\dot{\vec{v}} = F_L$

(1) $\rightsquigarrow \underline{\dot{\vec{v}}} = \frac{-eB}{mc} v\hat{r} = \underline{-\omega_c v\hat{r}}$

with $\omega_c = eB/mc$ the cyclotron frequency

$\rightsquigarrow P = \frac{2e^2}{3c^3}(\dot{\vec{v}})^2 = \frac{2e^2}{3c^3}\omega_c^2 v^2 = \frac{4e^2}{3c^3 m}\omega_c^2 \underbrace{\frac{m}{2}v^2}_{=E}$

(1) $= \frac{4e^2\omega_c^2}{3c^3 m} E$

b) $P = -\frac{dE}{dt} \rightsquigarrow \frac{dE}{dt} = -\frac{1}{\tau}E$ with $\frac{1}{\tau} = \frac{4e^2\omega_c^2}{3c^3 m}$

(1) $\rightsquigarrow E(t) = E(t=0)\,e^{-t/\tau}$

c) $\tau = \frac{3c^3 m}{4e^2}\frac{m^2c^2}{e^2B^2} = \frac{3c^5 m^3}{4e^4 B^2}$

electron: $e = 4.8\times10^{-10}$ esu $\quad 1T = 10^4 G$

$m = 9.1\times10^{-28}$ g

(1) $\rightsquigarrow \tau = \frac{3(3\times10^{10})^5(9.1\times10^{-28})^3}{4(4.8\times10^{-10})^4\times10^8} = 2.6$ s

4.3.3.) a) 1-d harm. osc:

$$m\ddot{x} = -k x$$

$$\ddot{x} = -\omega_0^2 x \qquad \omega_0^2 = k/m$$

$$\rightarrow (\ddot{x})^2 = \omega_0^4 x^2 = \omega_0^4 \frac{2}{k} \underbrace{\frac{k}{2} x^2}_{=V(x)} = \omega_0^4 \frac{2}{m\omega_0^2} V(x)$$

1

$$= \frac{2\omega_0^2}{m} V(x)$$

time average: $\overline{(\ddot{x})^2} = \frac{2\omega_0^2}{m}\overline{V(x)} = \frac{2\omega_0^2}{m}\frac{E}{2}$ by the virial theorem

$$\rightarrow \overline{(\ddot{x})^2} = \frac{\omega_0^2}{m} E$$

1

$$\rightarrow \underline{\underline{\overline{P}}} = \frac{2e^2}{3c^3}\overline{(\ddot{x})^2} = \underline{\underline{\frac{2e^2}{3c^3}\frac{\omega_0^2}{m} E}}$$

b) $\overline{P} = -\frac{dE}{dt} = \frac{1}{\tau} E$ with $\underline{\frac{1}{\tau} = \frac{2e^2}{3c^3}\frac{\omega_0^2}{m}}$

1

$$\rightarrow \underline{\underline{E(t) = E(t=0)\, e^{-t/\tau}}}$$

c) electron, $\omega_0 = 10^{15}\ s^{-1}$

1

$$\rightarrow \underline{\underline{\tau}} = \frac{3c^3}{2e^2}\frac{m}{\omega_0^2} = \frac{3(3\times 10^{10})^3}{2(4.8\times 10^{-10})^2}\frac{9.1\times 10^{-28}}{10^{30}}\ s = \underline{\underline{1.6\times 10^{-7}\ s}}$$