PHYS 623 Classical Electrodynamics S 2021

Problem Assignment # 14 04/29/2021 due 05/10/2021

4.3.4. **Classical atom**

Consider a classical electron in a circular orbit in a Coulomb potential, for which the virial theorem yields $\bar{V} = 2E$.

a) Assuming that the electron continues to move on a circle as it radiates, calculate the average power radiated as a function of E.

   hint: Note that the power is not a linear function of E, in contrast to the preceding two problems!

b) Show that the electron reaches the nucleus in a finite amount of time. For a hydrogen atom, calculate that time if the initial orbit had a radius $r_0 = 10^{-8}\,\text{cm}$.

(5 points)

4.3.5. **Absence of dipole radiation**

Show that a system of particles that all have to the same ratio of charge to mass and are not subject to any external forces cannot emit either electric or magnetic dipole radiation.

(3 points)

4.3.6. **Rotating dipole**

An electric dipole moment $\boldsymbol{d}$ rotates uniformly with angular velocity Ω in a plane. Find the radiated power per solid angle, and the total radiated power, averaged over one rotational period.

(5 points)

4.3.1.) a)

Coulomb potential: $V = \frac{-\alpha}{r}$

force $\vec{F} = \frac{-\alpha}{r^2}\hat{r} = m\dot{v}$

$$\to \dot{\vec{v}} = \frac{-\alpha}{mr^2}\hat{r}$$

$$\to \underline{(\dot{\vec{v}})^2} = \frac{\alpha^2}{m^2 r^4} = \underline{\frac{1}{\alpha^2 m^2}(V(r))^4}$$

~~kinetic~~ energy + virial theorem + circular orbit ($\to \overline{V^4} = \bar{V}^4$)

$$\to \overline{(\dot{\vec{v}})^2} = \frac{1}{\alpha^2 m^2}\overline{V^4} = \frac{1}{\alpha^2 m^2}16E^4$$

1

$$\to \underline{\underline{\bar{P}}} = \frac{2e^2}{3c^3}(\dot{\vec{v}})^2 = \frac{2e^2}{3c^3}\frac{16}{\alpha^2 m^2}E^4 = \underline{\underline{\frac{32e^2}{3c^3}\frac{1}{\alpha^2 m^2}E^4}}$$

b) $\bar{P} = -\frac{dE}{dt} = \Lambda E^4$ with $\Lambda = \frac{32e^2}{3c^3}\frac{1}{\alpha^2 m^2}$

1

$$\to \frac{dE}{E^4} = -\Lambda dt$$

$$\to -\frac{1}{3}\left(\frac{1}{E^3} - \frac{1}{E_0^3}\right) = -\Lambda t \quad \text{with } E_0 = E(t=0)$$

$$\to \frac{1}{E^3} = \frac{1}{E_0^3} + 3\Lambda t$$

1

$$\to \underline{E(t)} = \underline{\frac{E_0}{(1+3E_0^3\Lambda t)^{1/3}}} \quad \text{with } \underline{E_0 < 0}$$

$\to$ Particle hits the center (i.e., $E = -\infty$) at time

$$t^* = \frac{-1}{3E_0^3\Lambda}$$

Put $E_0 = \frac{1}{2} V_0 = -\frac{\alpha}{2r_0}$ with r_0 the starting radius

1 $\quad \rightarrow \underline{\underline{t^*}} = \frac{8r_0^3}{\cancel{3}\alpha^3}\,\frac{\cancel{3}c^3}{32e^2}\,\alpha^2 m^2 = \underline{\underline{\frac{c^3}{4e^2}\,\frac{m^2}{\alpha}\,r_0^3}}$

Hydrogen: $\quad \alpha = e^2 \qquad r_0 = 10^{-8}\,\text{cm}$

$$\rightarrow \underline{\underline{t^*}} = \frac{c^3}{4e^4}\, m^2 r_0^3 = \frac{(3\times10^{10})^3}{4\times(4.8\times10^{-10})^4}\,(9.1\times10^{-28})^2\times10^{-24}\,\text{s}$$

1 $\quad \underline{\underline{= 0.35\times10^{-10}\,\text{s}}}$

4.3.5.) Consider a set of charges e_α with mass m_α.
The electric dipole moment is

$$\vec{d} = \sum_\alpha e_\alpha \vec{x}_\alpha = \sum_\alpha \frac{e_\alpha}{m_\alpha} m_\alpha \vec{x}_\alpha$$

-> If $e_\alpha / m_\alpha = \text{const}$, then

1

$$\vec{d} = \text{const} \sum_\alpha m_\alpha \vec{x}_\alpha = \text{const.}\ \vec{X}(t)$$

with $\vec{X}(t)$ the center of mass. But $\vec{X}(t)$ moves uniformly -> $\ddot{\vec{d}}(t) \propto \ddot{\vec{X}}(t) = 0$ -> no electric dipole radiation

1

The magnetic dipole moment is

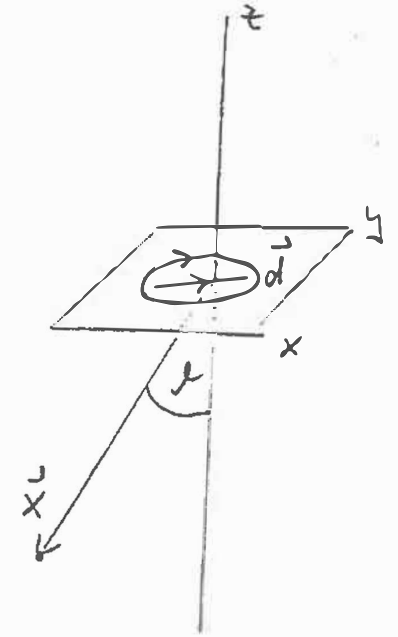
$$\vec{m} = \frac{1}{2c} \sum_\alpha e_\alpha \vec{x}_\alpha \times \vec{v}_\alpha = \frac{1}{2c} \sum_\alpha \frac{e_\alpha}{m_\alpha} \vec{x}_\alpha \times m\vec{v}_\alpha$$

$$= \text{const} \times \frac{1}{2c} \sum_\alpha \vec{x}_\alpha \times \vec{p}_\alpha = \text{const} \times \frac{1}{2c} \vec{L}(t)$$

with $\vec{L} = \sum_\alpha \vec{x}_\alpha \times \vec{p}_\alpha = \sum_\alpha \vec{x}_\alpha \times m_\alpha \vec{v}_\alpha$ the total angular momentum

But angular momentum is conserved -> $\dot{\vec{L}}(t) = 0$

1 -> $\ddot{\vec{m}}(t) = 0$ -> no magnetic dipole radiation



4.3.6.) let $\vec d$ lie in the x-y-plane

$\rightsquigarrow$ $d_x = d\cos\Omega t$

$d_y = d\sin\Omega t$ with $d = |\vec d|$.

(1) $\rightsquigarrow$ $\ddot{\vec d}(t) = \Omega^2 \vec d(t)$

let ϑ be the angle between the direction $\vec x$ of the observer and the z-axis $\rightsquigarrow$ $(\hat x \times \ddot{\vec d})^2 = \Omega^2 (\hat x \times \vec d)^2$

Consider

$$\hat x \times \vec d(t) = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix} \times \begin{pmatrix} d_x(t) \\ d_y(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -r\cos\vartheta\, d_y(t) \\ r\cos\vartheta\, d_x(t) \\ x d_y(t) - y d_x(t) \end{pmatrix}$$

$$\rightsquigarrow \underline{(\vec x \times \vec d(t))^2} = r^2\cos^2\vartheta\, \vec d^{\,2} + x^2 d_y^2(t) + y^2 d_x^2(t) - 2xy\, d_x(t) d_y(t)$$

$$= r^2\cos^2\vartheta\, \vec d^{\,2} + r^2\sin^2\vartheta\cos^2\varphi\, d^2\cos^2\Omega t + r^2\sin^2\vartheta\sin^2\varphi\, d^2\sin^2\Omega t - 2r^2\sin^2\vartheta\cos\varphi\sin\varphi\, d^2\cos\Omega t\sin\Omega t$$

(1)

Average over one rotation period: $T = 2\pi/\Omega$:

$$\overline{\cos^2\Omega t} = \frac{1}{T}\int_0^T dt\, \cos^2\frac{2\pi}{T}t = \frac{1}{2\pi}\int_0^{2\pi} dx\, \cos^2 x = \underline{\frac{1}{2}}$$

$$\overline{\sin^2\Omega t} = \underline{\frac{1}{2}}$$

$$\underline{\overline{\cos\Omega t\, \sin\Omega t} = 0}$$

$$\rightsquigarrow \overline{(\vec x \times \vec d(t))^2} = \left(r^2\cos^2\vartheta + \frac{1}{2} r^2 \sin^2\vartheta\right) d^2 = r^2\left(\cos^2\vartheta + \frac{1}{2} - \frac{1}{2}\cos^2\vartheta\right),$$

(1)

$$= \underline{\frac{1}{2} r^2 (1 + \cos^2\vartheta)}$$

ch 5 § 3.4 → The power per solid angle, averaged over one rotation period, is

$$\overline{dP/d\Omega} = \frac{1}{4\pi c^3}\,\overline{\left(\hat{x}\times\ddot{\vec{d}}(t)\right)^2} = \frac{\Omega^4}{4\pi c^3}\,\overline{\left(\hat{x}\times\vec{d}(t)\right)^2}$$

(1)
$$= \frac{\Omega^4 d^2}{4\pi c^3}\,\frac{1}{2}\left(1+\cos^2\vartheta\right) = \frac{\Omega^4 d^2}{8\pi c^3}\left(1+\cos^2\vartheta\right)$$

→ The total radiated power, averaged over one period, is

$$\overline{P} = \int d\Omega\;\overline{dP/d\Omega} = 2\pi\int_{-1}^{1} d(\cos\vartheta)\left(1+\cos^2\vartheta\right)\frac{\Omega^4 d^2}{8\pi c^3}$$

(1)
$$= \frac{\Omega^4 d^2}{4c^3}\,2\left(1+\frac{1}{3}\right) = \frac{2d^2}{3c^3}\,\Omega^4$$