Problem Assignment # 15 05/06/2021 due 05/20/2021

4.4.1. **Radiation by an accelerated point particle**

Show that the result of ch.4 §4.4 example (2) for the energy radiated by an accelerated point particle,

$$\frac{dU}{d\omega} = \frac{2}{3}\,\frac{e^2}{\pi c^3}\,|\dot{\boldsymbol{v}}(\omega)|^2$$

is consistent with the Larmor formula for the radiated power, ch.4 §3.3:

$$\mathcal{P}(t) = \frac{2e^2}{3c^3}\,\left(\dot{\boldsymbol{v}}(t)\right)^2$$

(2 points)

4.4.2. **Classical model of an atom**

Consider the classical model of a radiating atom from ch.4 §4.5; i.e., a damped harmonic oscillator with charge e and equation of motion

$$\ddot{y} = -\omega_0^2\,y - \gamma\,\dot{y}$$

Do NOT assume the damping to be small and solve the equation of motion exactly.

a) Assuming that the oscillator represents an electron, and that the spectrum is peaked somewhere in the range of visible light, use the approximate result from ch.4 §4.5 to show that one does not need to consider the case of the oscillator being overdamped.

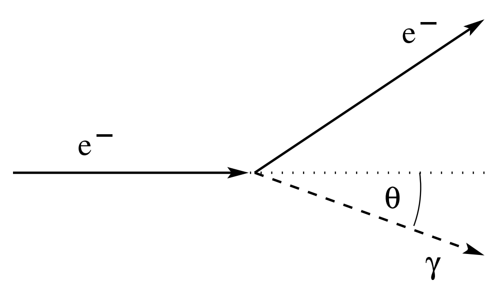
b) Calculate again the radiation spectrum $dU/d\omega$, and the total energy U radiated, exactly. Compare with the approximate result obtained in class. Show that the result self-consistently validates the assumption made in part a) above.

(8 points)

4.5.1. **Čerenkov radiation**

a) Consider a particle with rest mass m_0 that travels through a medium with index of refraction n. The particle emits a photon that moves at an angle θ with respect to the particle's initial trajectory. Use relativistic kinematics to express θ in terms of the particle's initial energy and momentum, the wave number of the photon, and n.

b) Show that to leading order for $c \to \infty$ one recovers the result of the nonrelativistic approximation used in ch. 4 §5.2.



c) The core of a https://www.youtube.com/watch?v=mgNwtepP-6Mwater-cooled reactor emits a 1.42 MeV electron that travels through the water ($n = 1.33$). Calculate θ for blue light ($\lambda = 4,000\,\AA$). How good or bad is the nonrelativistic approximation in this case?

(6 points)

4.4-1.) The Larmor formula implies for the total radiated energy

$$\underline{U} = \int dt\, P(t) = \underline{\frac{2e^2}{3c^3} \int dt\, (\dot{\vec{v}}(t))^2} \qquad (*)$$

whereas § 4.4 implies

$$\underline{U} = \int_0^\infty d\omega\, dU/d\omega = \int_0^\infty d\omega\, \frac{2e^2}{3\pi c^3} |\dot{\vec{v}}(\omega)|^2$$

$$= \underline{\frac{2e^2}{3c^3} \frac{1}{\pi} \int_0^\infty d\omega\, |\dot{\vec{v}}(\omega)|^2} \qquad (**)$$

$\underline{①}$

Now let $f(t) \in \mathbb{R}$ be a fct. of time, and

$f(\omega) = \int dt\, e^{i\omega t} f(t)$ its Fourier transfo.

Then $\underline{\int_0^\infty d\omega\, |f(\omega)|^2} = \int_0^\infty d\omega\, f(\omega)\, f^*(\omega)$

$$= \frac{1}{2} \int_{-\infty}^{\infty} d\omega \int dt\, e^{i\omega t} f(t) \int dt'\, e^{-i\omega t'} f(t')$$

$$= \frac{1}{2} \int dt\, dt'\, f(t) f(t') \underbrace{\int d\omega\, e^{i\omega(t-t')}}_{= 2\pi\delta(t-t')}$$

$$= \underline{\pi \int dt\, (f(t))^2}$$

$\underline{①}$ -> $\underline{\underline{(*) = (**)}}$

4.4.2.) a) Eq. of motion: $\ddot{y} + \omega_0^2 y + \gamma \dot{y} = 0$

ansatz: $y(t) = e^{rt} \rightsquigarrow r^2 + \gamma r + \omega_0^2 = 0$

(1) $\rightsquigarrow r = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)$

1st case: $\gamma < 2\omega_0$ underdamped

$$r = \pm i\omega_0 \sqrt{1-\gamma^2/4\omega_0^2} - \frac{1}{2}\gamma =: \pm i\tilde{\omega}_0 - \frac{1}{2}\gamma$$

$$\rightsquigarrow y(t) = c_1 e^{i\tilde{\omega}_0 t - \frac{1}{2}\gamma t} + c_2 e^{-i\tilde{\omega}_0 t - \frac{1}{2}\gamma t}$$

$$y(t=0) = c_1 + c_2 \overset{!}{=} a$$

$$\dot{y}(t=0) = c_1\left(i\tilde{\omega}_0 - \frac{1}{2}\gamma\right) + \left(-i\tilde{\omega}_0 - \frac{1}{2}\gamma\right)c_2$$

$$= -\frac{a}{2}\gamma + (c_1 - c_2)\, i\tilde{\omega}_0 \overset{!}{=} 0$$

$$\rightsquigarrow c_2 = c_1 - \frac{a}{2}\gamma \frac{1}{i\tilde{\omega}_0} = c_1 + ia\frac{\gamma}{2\tilde{\omega}_0}$$

$$= a - c_1$$

$$\rightsquigarrow c_1 = \frac{a}{2} - i\frac{a}{4}\frac{\gamma}{\tilde{\omega}_0}, \quad c_2 = \frac{a}{2} + i\frac{a}{4}\frac{\gamma}{\tilde{\omega}_0}$$

$$\rightsquigarrow y(t) = a\left[\cos\hat{\omega}_0 t + \frac{\gamma}{2\hat{\omega}_0}\sin\hat{\omega}_0 t\right] e^{-\gamma t/2}$$

$$\dot{y}(t) = a\left[-\hat{\omega}_0 \sin\hat{\omega}_0 t + \frac{\gamma}{2\cancel{\hat{\omega}_0}}\cancel{\hat{\omega}_0}\cos\hat{\omega}_0 t\right] e^{-\gamma t/2}$$

$$- a\frac{1}{2}\gamma\left[\cos\hat{\omega}_0 t + \frac{\gamma}{2\hat{\omega}_0}\sin\hat{\omega}_0 t\right] e^{-\gamma t/2}$$

$$= -a\hat{\omega}_0\left[1 + \left(\frac{\gamma}{2\hat{\omega}_0}\right)^2\right]\sin\hat{\omega}_0 t\, e^{-\gamma t/2}$$

(1) where $\hat{\omega}_0 = \omega_0\sqrt{1-(\gamma/2\omega_0)^2}$

2nd case: $\gamma = 2\omega_0$ critically damped

1 3rd case: $\gamma > 2\omega_0$ overdamped

cl 5 §4.5 → the first answer (to be confirmed self-consistently below) is $\gamma \simeq \frac{e^2\omega_0^2}{mc^3}$

→ $\frac{\gamma}{\omega_0} \simeq \frac{e^2\omega_0/c}{mc^2} = \frac{2\pi e^2/\lambda}{mc^2}$ with λ the wavelength of the radiation

$\simeq \frac{e^2/a_B}{mc^2} \frac{2\pi a_B}{\lambda}$ with a_B the Bohr radius

$\simeq \frac{10\,\text{eV}}{5\times10^5\,\text{eV}} \cdot \frac{10\,\text{Å}}{5{,}000\,\text{Å}} \simeq 4\times10^{-8}$ for electrons and visible light

1 → We always have the underdamped case, except for oscillators with a very large frequency ω_0.

→ Consider the underdamped case only.

b) Now we have $v(t) = \dot{y}(t) = a\tilde{\tilde{\omega}}_0 \sin \tilde{\omega}_0 t \, e^{-\gamma t/2}$

with $\tilde{\tilde{\omega}}_0 = \tilde{\omega}_0 \left[1 + (\gamma/2\tilde{\omega}_0)^2\right]$

This yields (see cl 5 §4.5)

1 $$v(\omega) = \frac{a\tilde{\tilde{\omega}}_0}{2} \left[\frac{1}{\omega - \tilde{\omega}_0 + i\gamma/2} - \frac{1}{\omega + \tilde{\omega}_0 + i\gamma/2}\right]$$

$$\rightarrow \frac{dU}{d\omega} = \frac{2e^2}{3\pi c^3}|\dot{v}(\omega)|^2 = \frac{e^2 a^2\tilde{\omega}_0^2\omega^2}{6\pi c^3}\left[\frac{1}{\omega-\tilde{\omega}_0+i\gamma/2}-\frac{1}{\omega+\tilde{\omega}_0+i\gamma/2}\right] \times\left[\frac{1}{\omega-\tilde{\omega}_0-i\gamma/2}-\frac{1}{\omega+\tilde{\omega}_0-i\gamma/2}\right]$$

$$= \frac{e^2a^2\tilde{\tilde{\omega}}_0^2\omega^2}{6\pi c^3}\left[\frac{1}{(\omega-\tilde{\omega}_0)^2+\gamma^2/4}+\frac{1}{(\omega+\tilde{\omega}_0)^2+\gamma^2/4}-2\,\mathrm{Re}\,\frac{1}{(\omega-\tilde{\omega}_0+i\gamma/2)(\omega+\tilde{\omega}_0-i\gamma/2)}\right]$$

$$= \frac{e^2a^2\tilde{\tilde{\omega}}_0^2\omega^2}{6\pi c^3}\left[\frac{1}{(\omega-\tilde{\omega}_0)^2+\gamma^2/4}+\frac{1}{(\omega+\tilde{\omega}_0)^2+\gamma^2/4}-2\frac{\omega^2-\tilde{\omega}_0^2+\gamma^2/4}{[(\omega-\tilde{\omega}_0)^2+\gamma^2/4][(\omega+\tilde{\omega}_0)^2+\frac{\gamma^2}{4}]}\right]$$

$$= \frac{e^2a^2\tilde{\tilde{\omega}}_0^2\omega^2}{6\pi c^3}\,\frac{\cancel{2\omega^2}+2\tilde{\omega}_0^2+\cancel{\gamma^2/2}-\cancel{2\omega^2}+2\tilde{\omega}_0^2-\cancel{\gamma^2/2}}{[(\omega-\tilde{\omega}_0)^2+\gamma^2/4][(\omega+\tilde{\omega}_0)^2+\gamma^2/4]}$$

$$= \frac{2}{3\pi}\frac{e^2a^2\tilde{\omega}_0^2\tilde{\tilde{\omega}}_0^2}{c^3}\frac{\omega^2}{[(\omega-\tilde{\omega}_0)^2+\gamma^2/4][(\omega+\tilde{\omega}_0)^2+\gamma^2/4]}$$

1

$$\rightarrow U = \int d\omega\,\frac{dU}{d\omega} =$$

$$= \frac{2}{3\pi}\frac{e^2a^2\tilde{\omega}_0^2\tilde{\tilde{\omega}}_0^2}{c^3}\frac{1}{\tilde{\omega}_0}\int dx\,\frac{x^2}{[(x-1)^2+\gamma^2/4\tilde{\omega}_0^2][(x+1)^2+\gamma^2/4\tilde{\omega}_0^2]}$$

$$= \frac{2}{3\cancel{\pi}}\frac{e^2a^2\tilde{\omega}_0\tilde{\tilde{\omega}}_0^2}{c^3}\frac{\pi}{2}\frac{2\tilde{\omega}_0}{\gamma} = \frac{2e^2a^2\tilde{\omega}_0^2\tilde{\tilde{\omega}}_0^2}{3c^3\gamma}$$

$$\tilde{\omega}_0^2\tilde{\tilde{\omega}}_0^2 = \tilde{\omega}_0^4[1+(\gamma/2\tilde{\omega}_0)^2] = \omega_0^4[1-(\gamma/2\omega_0)^2]^2\left[1+\frac{(\gamma/2\omega_0)^2}{1-(\gamma/2\omega_0)^2}\right] = \omega_0^4$$

$$= 2e^2a^2\omega_0^4/3c^3\gamma$$

1

compare with cl 4 §4.5 → the effects of the approximations we made in that calculation all cancel!

→ γ/ω_0 is indeed as small as estimated in part a). This justifies both considering the underdamped case only, and the approximations made in cl 5 §4.5

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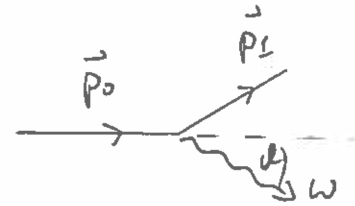
4.5.1 a). particle: rest mass: m_0

initial momentum, energy: $\vec{p}_0, E_0$



final momentum, energy: $\vec{p}_1, E_1$

energy-momentum relation: $E = \sqrt{m_0^2c^4 + p^2c^2}$

photon: frequency ω wave vector $\vec{k}$

momentum: $\vec{p}_\gamma = \hbar\vec{k}$, $p_\gamma = |\vec{p}_\gamma| = \hbar k = \hbar\omega/c_{eff}$

energy-momentum relation: $E_\gamma = \hbar\omega = c_{eff}\, p_\gamma$

medium: index of refraction n

speed of light: $c_{eff} = c/n$

1

energy conservation: $E_0 = E_1 + \hbar\omega$ (1)

momentum conservation: $\vec{p}_0 = \vec{p}_1 + \hbar\vec{k}$ (2)

(2) → $p_0^2 + \hbar^2k^2 - 2p_0 k\hbar\cos\vartheta = p_1^2 = E_1^2/c^2 - m_0^2c^2$ (*)

1 (1) → $E_1 = c\sqrt{m_0^2c^2 + p_0^2} - c_{eff}\hbar k$

→ $E_1^2 = c^2(m_0^2c^2 + p_0^2) + \hbar^2k^2c_{eff}^2 - 2c_{eff}\hbar k E_0$ (**)

(**) in (*) → $\cancel{p_0^2} + \hbar^2k^2 - 2p_0\hbar k\cos\vartheta = \cancel{m_0^2c^2} + \cancel{p_0^2} + \hbar^2k^2\frac{c_{eff}^2}{c^2} - 2\frac{\hbar k c_{eff}}{c^2}E_0 - \cancel{m_0^2c^2}$

→ $2p_0\hbar k\cos\vartheta = \hbar^2k^2\left(-\frac{c_{eff}^2}{c^2} + 1\right) + 2\hbar k\frac{c_{eff}}{c^2}E_0$

→ $2p_0\cos\vartheta = \hbar k(1 - 1/n^2) + 2E_0/cn$

→ $\boxed{\cos\vartheta = \frac{E_0}{cp_0n} + \frac{\hbar k}{p_0}\frac{n^2-1}{2n^2}}$

1

b) $p_0 = m v_0 = m_0 v_0 \gamma$ with $\gamma = \frac{1}{\sqrt{1-v_0^2/c^2}}$

$E_0 = mc^2 = m_0 c^2 \gamma$

$\rightarrow \underline{\underline{\cos\vartheta}} = \frac{\cancel{m_0} c^{2} \cancel{\gamma}}{c \cancel{m_0} v_0 \cancel{\gamma} n} + O(c^0) = \underline{\underline{\frac{c}{v_0 n}\left[1 + O(1/c)\right]}}$

(1) cf 5 § 5.2 ✓

c) $\underline{E_0 = 1.42\ \text{MeV}} = \sqrt{m_0^2c^4 + p_0^2c^2}$ $\underline{m_0c^2 = 0.51\ \text{MeV}}$ for electrons

$\rightarrow p_0^2 = \frac{1}{c^2}\left(E_0^2 - m_0^2c^4\right)$

$\rightarrow \underline{cp_0} = \sqrt{1.42^2 - 0.51^2}\ \text{MeV} = \underline{1.33\ \text{MeV}}$

$\rightarrow \underline{\frac{E_0}{cp_0 n}} = \frac{1.42}{1.33} \cdot \frac{1}{1.33} = \underline{0.81}$

$\underline{\frac{\hbar k}{p_0}} = \frac{2\pi}{\lambda}\frac{\hbar}{p_0} = \frac{2\cancel{\pi}}{4\times10^{3}\times10^{-8}} \frac{10^{-27}\times3\times10^{10}}{1.33\times10^{6}\times1.6\times10^{-12}} \approx 10^{-5}\times\frac{1}{4\times1.33\times1.6}$

$\underline{\approx 10^{-6}}$

(1) $\rightarrow$ The term of $O(1/c)$ is a negligible correction!

$\rightarrow \underline{\cos\vartheta = 0.81}$

(1) $\underline{\underline{\vartheta = 36^0}}$