Problem Assignment # 17 05/20/2021
(ctd from #15) due 05/27/2021

4.6.1. **Properties of Bessel functions** (not graded)

note: AS refers to the book by Abramowitz and Stegun (which now goes by F.W.J. Olver et al, see the link on the web page)

a) Starting from the integral representation of the Bessel function J_m (see, e.g., AS 9.1.22)

$$J_n(x) = \int_0^\pi \frac{d\phi}{\pi}\, \cos(x \sin\phi - n\phi)$$

show that $J_{2n}(x)$ can be written as

$$J_{2n}(x) = \int_0^\pi \frac{d\phi}{\pi}\, \cos(x \sin(\phi/2))\, \cos(n\phi)$$

b) Show that for $n \gg 1$, $\beta \lessapprox 1$,

$$J'_n(\beta n) \approx \begin{cases} \frac{2^{2/3}}{3^{1/3}\Gamma(1/3)}\, n^{-2/3} & \text{for } 1 \ll n \ll \gamma^3 \\ \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sqrt{n\gamma}}\, e^{-n/3\gamma^3} & \text{for } n \gg \gamma^3 \end{cases}$$

where $J'_n(x)$ is the derivative of $J_n(x)$ with respect to its argument, and $\gamma = 1/\sqrt{1-\beta^2}$.

note: This can be shown by asymptotic analysis, starting from the integral representation of the Bessel function, but this is quite involved. For our present purposes, start with tabulated asymptotic expansions, e.g., AS 9.3.35 and 9.3.43, and take it from there.

c) Show that the Bessel function itself has the asymptotic behavior

$$J_n(\beta n) = \begin{cases} \frac{1}{3^{2/3}\Gamma(2/3)}\, (2/n)^{1/3} & \text{for } 1 \ll n \ll \gamma^3 \\ \frac{1}{\sqrt{2\pi}}\, \sqrt{\gamma/n}\, e^{-n/3\gamma^3} & \text{for } n \gg \gamma^3 \end{cases}$$

(6 points)

4.6.2. **Synchrotron radiation** (to be continued next week)

a) Starting from the expression for the radiated power in ch. 4 §6.2 lemma 1, integrate over the angles to show that the power spectrum of synchrotron radiation can be written

$$\frac{dP}{d\omega} = \frac{\omega}{2\pi}\, \frac{e^2}{R} \int d\tau\, e^{i\omega\tau}\, f(\omega_0\tau)$$

Determine the function f and show that it is 2π-periodic.

b) Expand the periodic function f from part a) in a Fourier series and perform the τ-integration to show that the power spectrum takes the form

$$\frac{dP}{d\omega} = \sum_{m=1}^{\infty} \delta(\omega - m\,\omega_0)\, P_m$$

with P_m expressed in terms of an integral.

(9 points)

4.6.) a) AS 9.1.22 ⇝

$$\underline{J_n(x)} = \int_0^\pi \frac{d\varphi}{\pi} \cos(x\sin\varphi - n\varphi)$$

$$= \int_0^{\pi/2} \frac{d\varphi}{\pi} \cos(x\sin\varphi - n\varphi) + \int_{\pi/2}^{\pi} \frac{d\varphi}{\pi} \cos(x\sin\varphi - n\varphi)$$

$$\overset{\phi = \pi - \varphi}{=} \int_0^{\pi/2} \frac{d\varphi}{\pi} \cos(x\sin\varphi - n\varphi) - \int_{\pi/2}^{0} \frac{d\phi}{\pi} \cos\left(x\sin(\pi-\phi) + n\phi - n\pi\right)$$

$$= \underline{\int_0^{\pi/2} \frac{d\varphi}{\pi} \left[\cos(x\sin\varphi - n\varphi) + \cos(x\sin\varphi + n\varphi - n\pi)\right]}$$

$$\leadsto \underline{\underline{J_{2n}(x)}} = \int_0^{\pi/2} \frac{d\varphi}{\pi} \left[\cos(x\sin\varphi - 2n\varphi) + \cos(x\sin\varphi + 2n\varphi)\right]$$

$$= \int_0^{\pi/2} \frac{d\varphi}{\pi} \left[\cos(x\sin\varphi)\cos 2n\varphi + \cancel{\sin(x\sin\varphi)\sin 2n\varphi} + \cos(x\sin\varphi)\cos 2n\varphi - \cancel{\sin(x\sin\varphi)\sin 2n\varphi}\right]$$

$$= \underline{\underline{\int_0^{\pi} \frac{d\varphi}{\pi} \cos\left(x\sin\frac{\varphi}{2}\right)\cos n\varphi}}$$

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b) AS 9.3.43 ⇝

$$\underline{\underline{J_n'(\lambda n)}} = \underline{-\frac{2}{\lambda}\left(\frac{1-\lambda^2}{4\zeta}\right)^{1/4} \frac{Ai'(n^{2/3}\zeta)}{n^{2/3}}\, d_0(\zeta) + O(n^{-4/3})}$$

where $\zeta = \left(\frac{3}{2}\left[\ln\frac{1+\sqrt{1-\lambda^2}}{\lambda} - \sqrt{1-\lambda^2}\right]\right)^{2/3}$ from AS 9.3.38

and $d_0(\zeta) = 1$ from AS 9.3.46

Note that the Ai' term in 9.3.43 gives the largest contribution for $n \gg 1$, not the Ai term.

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Define $\gamma = \frac{1}{\sqrt{1-\beta^2}} \to \beta = \sqrt{1-1/\gamma^2}$

and consider $\beta \to 1^- \Leftrightarrow \gamma \to \infty$

$$\to \underline{\zeta} = \left(\frac{3}{2}\right)^{2/3}\left[\ln\frac{1+1/\gamma}{\sqrt{1-1/\gamma^2}} - \frac{1}{\gamma}\right]^{2/3} \simeq \left(\frac{3}{2}\right)^{2/3}\frac{1}{(3\gamma^3)^{2/3}}\left[1+O(\gamma^{-1})\right]$$

$$= \underline{\frac{1}{2^{2/3}\gamma^2} + O(\gamma^{-4})} \quad \text{and} \quad \underline{\beta = 1+O(1/\gamma^2)}$$

$$\to \underline{J_n'(\beta n)} \simeq \frac{-2}{n^{2/3}}\left(\frac{1}{\cancel{\gamma^2}}\frac{1}{4}2^{2/3}\cancel{\gamma^2}\right)^{1/4} Ai'(n^{2/3}\zeta) = \underline{\frac{-2^{2/3}}{n^{2/3}} Ai'(n^{2/3}\zeta)}$$

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Now $\underline{Ai'(x \to 0)} \simeq \underline{\frac{-1}{3^{1/3}\Gamma(1/3)}} + O(x^2)$

$\to$ For $n^{2/3}\zeta \ll 1 \Leftrightarrow n^{2/3}/\gamma^2 \ll 1 \Leftrightarrow n/\gamma^3 \ll 1$

I have

$$\underline{J_n'(\beta n) \simeq \left(\frac{2}{n}\right)^{2/3}\frac{1}{3^{1/3}\Gamma(1/3)}} \quad \text{for} \quad \underline{1 \ll n \ll} \gamma^3$$

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While $\underline{Ai'(x \to \infty)} \simeq \underline{\frac{-x^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}\left[1+O(1/x)\right]}$

$\to$ For $n/\gamma^3 \gg 1$ I have

$$\underline{J_n'(\beta n)} \simeq \left(\frac{2}{n}\right)^{2/3}\frac{1}{2\sqrt{\pi}}\left(\frac{n^{2/3}}{2^{2/3}\gamma^2}\right)^{1/4} e^{-\frac{2}{3}\left(n^{2/3}/2^{2/3}\gamma^2\right)^{3/2}}$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{\gamma}}\left(\frac{2}{n}\right)^{1/2} e^{-\frac{2}{3}\left(\frac{n}{2}\right)/\gamma^3}$$

$$= \underline{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\gamma n}} e^{-n/3\gamma^3}} \quad \text{for} \quad \underline{n \gg \gamma^3}$$

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c) Some for $J_n(\Lambda n)$: AS 9.3.35 $\to$

$$J_n(\Lambda n) \simeq \left(\frac{4\zeta}{1-\Lambda^2}\right)^{1/4} \frac{1}{n^{1/3}}\, Ai(n^{2/3}\zeta)\, a_0(\zeta) + \dots \quad \text{with } a_0(\zeta = 1)$$

$$= \left(\cancel{\gamma^2}\frac{4}{2^{2/3}\cancel{\gamma^2}}\right)^{1/4} \frac{1}{n^{1/3}}\, Ai(n^{2/3}\zeta)$$

$$= \left(\frac{2}{n}\right)^{1/3} Ai(n^{2/3}\zeta)$$

Now $Ai(x \to 0) \overset{h}{=} \dfrac{1}{3^{2/3}\Gamma(2/3)} + O(x)$

$$\to \quad J_n(\Lambda n) \simeq \frac{1}{3^{2/3}\Gamma(2/3)}\left(\frac{2}{n}\right)^{1/3} \quad \text{for } 1 \ll n \ll \gamma^3$$

and $Ai(x \to \infty) \overset{h}{=} \dfrac{1}{2\sqrt{\pi}}\dfrac{1}{x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}}\left[1 + O(1/x)\right]$

$$\to \quad J_n(\Lambda n) \simeq \cancel{\left(\frac{2}{n}\right)^{1/3}} \frac{1}{\cancel{2}\sqrt{\pi}} \left(\frac{2^{2/3}\gamma^2}{\cancel{n^{2/3}}}\right)^{1/4} e^{-n/3\gamma^3}$$

$$= \frac{1}{\cancel{2}\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/2}\sqrt{\gamma}\; e^{-n/3\gamma^3}$$

(1) $$= \frac{1}{\sqrt{2\pi}}\sqrt{\gamma/n}\; e^{-n/3\gamma^3} \quad \text{for } n \gg \gamma^3$$

4.6.2) a) d4/J6.2 ->

$$\frac{d^2 P}{d\omega\, d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c^2} \int d\tau\, e^{i\omega\tau} \left[\vec{v}(T+\tau/2)\cdot\vec{v}(T-\tau/2) - c^2\right] e^{-i\frac{\omega}{c}\hat{x}\cdot[\vec{y}(T+\frac{\tau}{2}) - \vec{y}(T-\frac{\tau}{2})]}$$

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$$\rightarrow \frac{dP}{d\omega} = \frac{\omega^2 e^2}{4\pi^2 c^2} \int d\tau\, e^{i\omega\tau} \left[\vec{v}(T+\tau/2)\cdot\vec{v}(T-\tau/2) - c^2\right] \int d\Omega\, e^{-i\frac{\omega}{c}\hat{x}\cdot[\vec{y}(T+\frac{\tau}{2}) - \vec{y}(T-\frac{\tau}{2})]}$$

Consider $\vec{s} = \vec{y}(T+\tau/2) - \vec{y}(T-\tau/2)$

$$\rightarrow |\vec{s}| = \Big[R^2 + R^2 - 2R^2 \cos(\omega_0 T + \omega_0\tau/2)\cos(\omega_0 T - \omega_0\tau/2) - 2R^2 \sin(\omega_0 T + \omega_0\tau/2)\sin(\omega_0 T - \omega_0\tau/2)\Big]^{1/2}$$

$$= \sqrt{2}R\Big[1 - \left(\cos\omega_0 T\cos\tfrac{\omega_0\tau}{2} - \sin\omega_0 T\sin\tfrac{\omega_0\tau}{2}\right)\left(\cos\omega_0 T\cos\tfrac{\omega_0\tau}{2} + \sin\omega_0 T\sin\tfrac{\omega_0\tau}{2}\right) - \left(\sin\omega_0 T\cos\tfrac{\omega_0\tau}{2} + \cos\omega_0 T\sin\tfrac{\omega_0\tau}{2}\right)\left(\sin\omega_0 T\cos\tfrac{\omega_0\tau}{2} - \cos\omega_0 T\sin\tfrac{\omega_0\tau}{2}\right)$$

$$= \sqrt{2}R\Big[1 - \cos^2\omega_0 T\cos^2\tfrac{\omega_0\tau}{2} + \sin^2\omega_0 T\sin^2\tfrac{\omega_0\tau}{2} - \sin^2\omega_0 T\cos^2\tfrac{\omega_0\tau}{2} + \cos^2\omega_0 T\sin^2\tfrac{\omega_0\tau}{2}$$

$$= \sqrt{2}R\left[1 - \cos^2\tfrac{\omega_0\tau}{2} + \sin^2\tfrac{\omega_0\tau}{2}\right]^{1/2} = \sqrt{2}R\left[1 - \cos\omega_0\tau\right]^{1/2}$$

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$$= \sqrt{2}R\sqrt{2}\left|\sin\tfrac{\omega_0\tau}{2}\right| = 2R\left|\sin\tfrac{\omega_0\tau}{2}\right|$$

$$\rightarrow \int d\Omega\, e^{-i\frac{\omega}{c}\hat{x}\cdot\vec{s}} = 2\pi\int_{-1}^{1} d\eta\, e^{-i\frac{\omega}{c}|\vec{s}|\eta} = 2\pi\frac{ic}{\omega}\frac{1}{|\vec{s}|}\left[e^{-i\frac{\omega}{c}|\vec{s}|} - e^{i\frac{\omega}{c}|\vec{s}|}\right]$$

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$$= 2\pi\frac{ic}{\omega}\frac{1}{|\vec{s}|}(-2i)\sin\left(\frac{\omega}{c}|\vec{s}|\right) = 4\pi\frac{\sin(\omega|\vec{s}|/c)}{\omega|\vec{s}|/c}$$

Further, ->

$$\vec{v}(T+\tau/2)\cdot\vec{v}(T-\tau/2) = v^2\Big[\sin\left(\omega_0 T + \tfrac{\omega_0\tau}{2}\right)\sin\left(\omega_0 T - \tfrac{\omega_0\tau}{2}\right) + \cos\left(\omega_0 T + \tfrac{\omega_0\tau}{2}\right)\cos\left(\omega_0 T + \tfrac{\omega_0\tau}{2}\right)\Big]$$

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$$= v^2\cos\omega_0\tau$$

$$\to \frac{dP}{d\omega} = \frac{\omega^2 e^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau} \left[v^2 \cos\omega_0\tau - c^2\right] 4\pi \frac{\sin\left(\frac{\omega}{c} 2R \sin\frac{\omega_0\tau}{2}\right)}{\frac{\omega}{c} 2R \sin\frac{\omega_0\tau}{2}}$$

$$= \frac{\omega}{2\pi} \frac{e^2}{R} \int d\tau\, e^{i\omega\tau} \left[\beta^2 \cos\omega_0\tau - 1\right] \frac{\sin\left(\frac{\omega}{c} 2R \sin\frac{\omega_0\tau}{2}\right)}{\sin\frac{\omega_0\tau}{2}}$$

$$=: \frac{\omega}{2\pi} \frac{e^2}{R} \int d\tau\, e^{i\omega\tau} f(\omega_0\tau)$$

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where

$$f(x) = \left(\beta^2 \cos x - 1\right) \frac{\sin\left(\frac{\omega}{c} 2R \sin\frac{x}{2}\right)}{\sin(x/2)}$$

Now,

$$\sin\left(\frac{x+2\pi}{2}\right) = \sin\left(\frac{x}{2}+\pi\right) = -\sin\frac{x}{2}$$

$$\sin(-x) = -\sin x$$

$$\cos(x+2\pi) = \cos x$$

$$\to f(x+2\pi) = f(x)$$

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b) Write $f(\omega_0\tau)$ as a Fourier series,

$$f(\omega_0\tau) = \sum_{m=-\infty}^{\infty} e^{im\omega_0\tau} f_m$$

with coefficients

$$f_m = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{-imx} f(x) = f_{-m}^* \quad \text{since } f(x) \in \mathbb{R}$$

$$\to \frac{dP}{d\omega} = \frac{\omega}{2\pi} \frac{e^2}{R} \int d\tau\, e^{i\omega\tau} \sum_m e^{im\omega_0\tau} f_m$$

$$= \frac{\omega}{\cancel{2\pi}} \frac{e^2}{R} \sum_m f_m \cancel{2\pi}\, \delta(\omega + m\omega_0)$$

$$= \frac{\omega e^2}{R} \sum_m \delta(\omega - m\omega_0) f_{-m} = \frac{\omega e^2}{R} \sum_{m=1}^{\infty} \left[\delta(\omega - m\omega_0) f_{-m} + \delta(\omega + m\omega_0) f_m\right]$$

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Note $f(-x) = f(x) \to f_{-m} = f_m^* = f_m$

$\rightsquigarrow$ $\mathcal{P}(\omega) = \mathcal{P}(-\omega)$

$\rightsquigarrow$ Restrict ourselves to $\omega > 0$ w.l.g.

$$\frac{d\mathcal{P}}{d\omega} = \frac{\omega e^2}{R} \sum_{n=1}^{\infty} \delta(\omega - n\omega_0) f_n$$

$$= \sum_{n=1}^{\infty} \delta(\omega - n\omega_0) P_n$$

(1)

where

$$\underline{\underline{P_n}} = \frac{e^2\omega}{R} f_n = \frac{e^2\omega_0}{R} n \int \frac{dx}{2\pi} e^{-inx} (\Lambda^2 \cos x - 1) \frac{\sin\left(2n\frac{\omega_0 R}{c}\sin\frac{x}{2}\right)}{\sin(x/2)}$$

$\omega_0 R/c = \Lambda \downarrow$

$$= \frac{e^2\omega_0}{R} n \int_0^{2\pi} \frac{dx}{2\pi} e^{-inx} \left[\Lambda^2\left(1 - 2\sin^2\frac{x}{2}\right) - 1\right) \frac{\sin\left(2n\Lambda\sin\frac{x}{2}\right)}{\sin(x/2)}$$

$$\int^{2\pi} dx\, e^{-inx} g(x) = \quad \text{with } g(2\pi - x) = g(x)$$

$$= \int_0^{\pi} dx\, e^{-inx} g(x) + \int_{\pi}^{2\pi} dx\, e^{-inx} g(x)$$

$$\overset{y = 2\pi - x}{=} \int_0^{\pi} dx\, e^{-inx} g(x) + \int_0^{\pi} dy\, e^{iny} g(2\pi - y) = \int_0^{\pi} dx\, 2\cos nx\, g(x)$$

$$= \frac{e^2\omega_0}{R} n \int_0^{\pi} \frac{dx}{\pi} \cos(nx) \left[\Lambda^2 - 1 - 2\Lambda^2 \sin^2\frac{x}{2}\right] \frac{\sin\left(2n\Lambda\sin\frac{x}{2}\right)}{\sin(x/2)}$$

(1)