PHYS 623 Classical Electrodynamics S 2021

Problem Assignment # 18 05/27/2021

due 06/03/2021

4.6.2. **Synchrotron radiation** (continued)

c) Use the integral representation of the Bessel function J_{2m} from Problem 4.6.1 to show that P_m, the power radiated into the m-th harmonic, can be expressed in terms of Bessel functions as

$$P_m = \frac{e^2}{R}\, m\, \omega_0 \left[2\beta^2\, J'_{2m}(2m\beta) - (1-\beta^2) \int_0^{2m\beta} dx\, J_{2m}(x) \right]$$

where $\beta = v/c$ and J'_{2m} is the derivative of J_{2m} with respect to its argument.

note: One can also obtain this by integrating the final result from ch. 4 §6.2 over the angles, but that's harder.

d) Show that for $\beta \approx 1$ the power peaks at $m_{\rm max} \propto \gamma^3$, where $\gamma = 1/\sqrt{1-\beta^2}$.

hint: Analyze the J' contribution in detail and do what you can on the second term.

e) Estimate the peak frequency of the power spectrum and the corresponding wave length for the Advanced Light Source (1.9 GeV electrons in a circular orbit with radius $R \approx 20$m), and for a typical radio galaxy (5 GeV electrons in a field $B \approx 5\mu$G).

(7 points)

f) **This part is optional and meant for people who would like to gain a deeper understanding of the power distribution.** Work through Jackson ch. 14.6 to derive his result (14.84) for the angular distribution of the synchrotron radiation. Start with the expression for the power spectrum in the parallel-polarization state in § 6.3 and integrate over T to undo the Wigner structure. Then follow Jackson's logic and approximations, taking into account the difference between his coordinate system and ours, and also the different zeros of time. Repeat this for the perpendicular polarization, then add up the two and integrate over the frequency to obtain the angular distribution.

note: Note that Jackson's approximations are valid only for large γ and small angles about the orbital plane, and also involve a large-frequency approximation. Coming up with a *complete* expression for the angular distribution for all angles that captures both the ultrarelativistic and nonrelativistic limits is remarkably difficult.

4.7.1. **Scattering by a dielectric sphere** (not graded)

a) Argue on general grounds that the dipole moment of a dielectric sphere (radius a, dielectric constant ϵ) subject to an external electric field $\boldsymbol{E}_{\rm ext}$ is given by

$$\boldsymbol{d} = f(\epsilon)\, a^3\, \boldsymbol{E}_{\rm ext}\ ,$$

where the function f has the properties $f(\epsilon \to 1) = 0$, $f(\epsilon \to \infty) = {\rm const.}$ (We will determine $f(\epsilon)$ explicitly next week, see Problem 49.)

b) Find the scattering cross section for radiation with wavelength $\lambda \gg a$ scattered by the sphere.

(6 points)

4.6.2) Continued

c) Problem 4.6.1 ->

$$J_{2m}(y) = \int_0^\pi \frac{dx}{\pi} \cos(mx) \cos\left(y \sin\frac{x}{2}\right)$$

$$\Rightarrow \underline{J_{2m}'(y) = -\int_0^\pi \frac{dx}{\pi} \cos mx \sin\frac{x}{2} \sin\left(y \sin\frac{x}{2}\right)}$$

and $\underline{\int_0^y dz\, J_{2m}(z)} = \underline{\int_0^\pi \frac{dx}{\pi} \cos mx \frac{1}{\sin\frac{x}{2}} \sin\left(y \sin\frac{x}{2}\right)}$

$$\rightarrow \underline{\underline{P_m}} = \frac{e^2\omega_0}{R} m\Big\{ -2\beta^2 \int_0^\pi \frac{dx}{\pi} \cos(mx) \sin\frac{x}{2} \sin\Big(2m\beta \sin\frac{x}{2}\Big) - (1-\beta^2)\int_0^\pi \frac{dx}{\pi}\cos(mx) \frac{\sin(2m\beta\sin\frac{x}{2})}{\sin(x/2)} \Big\}$$

① $$= \underline{\underline{\frac{e^2\omega_0}{R} m \Big[2\beta^2 J'_{2m}(2m\beta) - (1-\beta^2)\int_0^{2m\beta} dx\, J_{2m}(x)\Big]}}$$

d) Probl 4.6.1 → For $m \gg 1$, $\beta \lesssim 1$

$$\underline{m J'_{2m}(2m\beta) \propto \begin{cases} m^{1/3} & \text{for } m \ll \gamma^3 \\ \sqrt{m}\, e^{-2m/3\gamma^3} & \text{for } m \gg \gamma^3 \end{cases}}$$

① → This contribution to P_m peaks around $m = m_{max} \simeq \gamma^3$

Now consider the second contribution. Define

$$f_m(\beta) = \int_0^{2m\beta} dx\, J_{2m}(x)$$

$$\rightarrow \frac{d}{d\beta} f_m(\beta) = 2m J_{2m}(2m\beta) \overset{\text{Probl 55}}{\propto} \begin{cases} m^{2/3} & \text{for } m \ll \gamma^3 \\ \sqrt{m\gamma}\, e^{-2m/3\gamma^3} & \text{for } m \gg \gamma^3 \end{cases}$$

I need to integrate this over β to obtain $f_m(\beta)$, but this cannot change the exponential dropoff for $m \gg \gamma^3$

① → I expect the second contribution to P_m to be qualitatively similar to the first one, but qualitatively smaller because of the $(1-\beta^2) = 1/\gamma^2$ prefactor. Check this numerically:

p-536-29-610

Check this numericallly :

```
Jp[m_, x_] := (BesselJ[m - 1, x] - BesselJ[m + 1, x]) / 2
J[m_, x_] := BesselJ[m, x]
```

```
beta = 0.99
gamma = 1 / Sqrt[1 - beta^2]
gamma^3
```

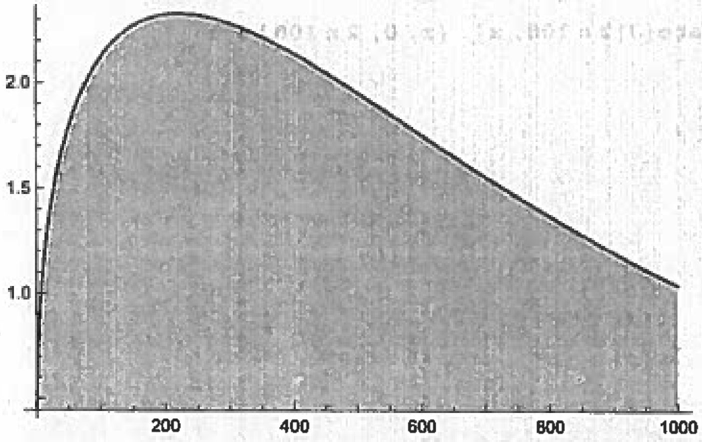
```
0.99

7.08881

356.222
```

```
beta = 0.99
gamma = 1 / Sqrt[1 - beta^2]
gamma^3
P[m_] :=
 m (2 beta^2 Jp[2 m, 2 m beta] - (1 - beta^2) NIntegrate[J[2 m, x], {x, 0, 2 m beta}])
DiscretePlot[P[m], {m, 1, 500}]
```

```
0.99

7.08881

356.222

$Aborted
```

Here is the first contribution to the spectrum :

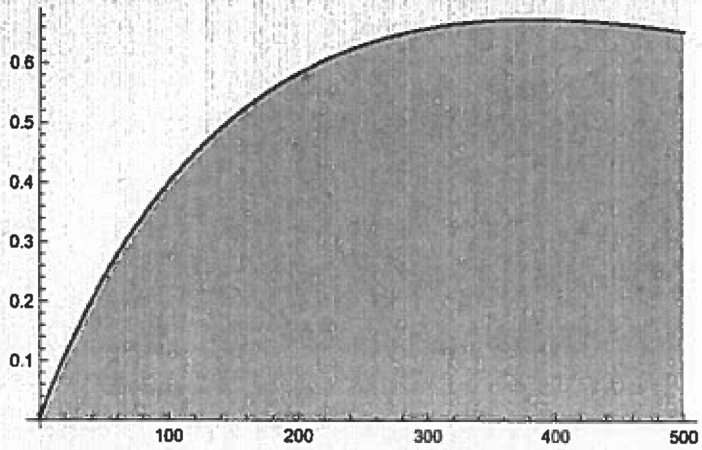
```
P1[m_] := 2 m beta^2 Jp[2 m, 2 m beta]
DiscretePlot[P1[m], {m, 1, 1000}]
```



It peaks for m between 200 and 250, where m_max = 356, so that ' s consistent. Now calculate the second contribution :

p-456-2-46

```
DiscretePlot[m (1 - beta^2) NIntegrate[J[2 m, x], {x, 0, 2 m beta}], {m, 1, 500}]
```



So this peaks a bit later than the first contribution, but it ' s always smaller :

```
Do[
 Print[P1[n 100]],
 {n, 5, 10}]
```

```
1.95462

1.75526

1.55763

1.37012

1.197

1.04009
```

```
Do[
 Print[n 100 (1 - beta^2) NIntegrate[J[2 n 100, x], {x, 0, 2 n 100 beta}]],
 {n, 5, 10}]
```

```
0.657352

0.617884

0.56815

0.514239

0.459964

0.407651
```

①

e) **ALS:** $E = \gamma mc^2 = \gamma \times 0.51 \text{ MeV} = 1.9 \times 10^3 \text{ GeV}$

$\rightsquigarrow \gamma \simeq 3{,}725 \quad (\rightsquigarrow v/c = 0.99999996\ldots\ !)$

$\rightsquigarrow \omega_0 = \frac{v}{R} = \frac{c}{R}\sqrt{1-1/\gamma^2} = 1.5 \times 10^7 \text{ Hz}$

$\rightsquigarrow \underline{\underline{\omega_{max}}} \simeq \omega_0 \gamma^3 = \underline{\underline{8 \times 10^{17} \text{ Hz}}}$

① $\rightsquigarrow \underline{\underline{\lambda_{max}}} = \frac{2\pi}{k_{max}} = \frac{2\pi c}{\omega_{max}} \simeq 2.4 \times 10^{-7} \text{ cm} \simeq \underline{\underline{24 \text{ Å}}}$

$\underline{\underline{\text{X-rays}}}$

galaxy: $\gamma = 5 \times 10^3 / 0.51 \simeq 10^4$

$\rightsquigarrow \omega_0 = \frac{ecB}{mc^2\gamma} = \frac{4.8 \times 10^{-10} \times 5 \times 10^{-6}}{9.1 \times 10^{-28} \times 3 \times 10^{10} \times 10^4} \text{ Hz} \simeq 10^{-2} \text{ Hz}$

$\rightsquigarrow \underline{\underline{\omega_{max}}} \simeq \omega_0 \gamma^3 \simeq 10^{-2} \times 10^{12} \text{ Hz} = \underline{\underline{10^{10} \text{ Hz}}}$

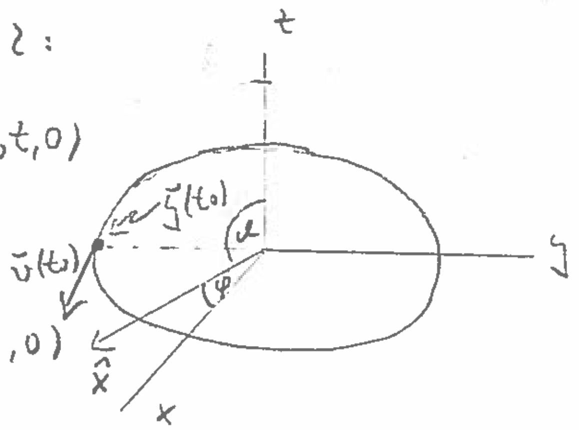
① $\rightsquigarrow \underline{\underline{\lambda_{max}}} = \frac{2\pi c}{\omega_{max}} \simeq \underline{\underline{19 \text{ cm}}} \qquad \underline{\underline{\text{radio waves}}}$

f) Consider the expression for the parallel polarized power spectrum from ill4 §6.4 and integrate over T (to undo the Wigner structure)

$$\underline{\underline{\left(\frac{d^2U}{d\omega\, d\Omega}\right)_{\parallel}}} = \int dT \left(\frac{d^2P(T)}{d\omega\, d\Omega}\right)_{\parallel} = \frac{\omega^2 e^2}{4\pi^2 c^3}\int dT \int d\tau\, e^{i\omega\tau}\, v_y(T+\tau/2)\, v_y(T-\tau/2) \times e^{-i\frac{\omega}{c}\hat{x}\cdot[\vec{\xi}(T+\tau/2)-\vec{\xi}(T-\tau/2)]}$$

$$= \frac{\omega^2 e^2}{4\pi^2 c^3}\int dt\, e^{i\omega t}\, v_y(t)\, e^{-i\frac{\omega}{c}\hat{x}\cdot\vec{\xi}(t)} \int dt'\, e^{-i\omega t'}\, v_y(t')\, e^{i\frac{\omega}{c}\hat{x}\cdot\vec{\xi}(t')}$$

Now consider the geometry from §6.2:



The orbit is $\vec{\xi}(t) = R(\cos\omega_0 t, \sin\omega_0 t, 0)$

and the velocity of the particle is

$$\vec{v}(t) = v(-\sin\omega_0 t, \cos\omega_0 t, 0)$$

and the radius vector to the observer is $\hat{x} = (\sin\vartheta, 0, \cos\vartheta)$

We know from the qualitative discussion in §6.2 that for $v \lesssim c$ the radiation will be

(i) confined to a narrow angle $\varphi = \frac{\pi}{2} - \vartheta$ about the orbital plane, and

(ii) observable only during a short time during which $\vec{v}$ points almost directly at the observer.

$\rightarrow$ Define $t_0 := -\pi/2\omega_0$ $\rightarrow$ $\vec{\xi}(t_0) = R(0,-1,0)$

$\vec{v}(t_0) = v\,(1,0,0)$

→ If we define φ and τ by $\quad \vartheta = \frac{\pi}{2} - \varphi$

$$t = t_0 + \tau$$

then we can confine ourselves to small values of φ and τ, provided $\beta = v/c \simeq 1$, or $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \gg 1$

(*) → We need

$$v_y(t)\, e^{i\omega t - i\frac{\omega}{c}\hat{x}\cdot\vec{\xi}(t)}$$

Now,

$$\underline{\hat{x}\cdot\vec{\xi}(t)} = R \sin\vartheta \cos\omega_0 t = R \sin\left(\tfrac{\pi}{2}-\varphi\right)\cos\left(-\tfrac{\pi}{2}+\omega_0\tau\right)$$

$$= \underline{R\cos\varphi \sin\omega_0\tau}$$

$$\rightarrow \underline{i\omega t - i\frac{\omega}{c}\hat{x}\cdot\vec{\xi}(t)} = i\omega t_0 + i\omega\left[\tau - \frac{R}{c}\cos\varphi\sin\omega_0\tau\right]$$

$$= i\omega t_0 + i\omega\left[\tau - \frac{R}{c}\left(1-\tfrac{1}{2}\varphi^2+O(\varphi^4)\right)\omega_0\tau\left(1-\tfrac{1}{6}\omega_0^2\tau^2+O(\tau^4)\right)\right]$$

$$= i\omega t_0 + i\omega\left[\tau\left(1-\frac{R\omega_0}{c}\right) + \frac{R\omega_0}{2c}\varphi^2\tau + \frac{R\omega_0^3}{6c}\tau^3 + O(\tau^3\varphi^2)\right]$$

$$\boxed{1-\frac{R\omega_0}{c} \underset{R\omega_0 = v}{\overset{\wedge}{=}} 1-\frac{v}{c} \simeq \frac{1}{2}\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right) = \frac{1}{2}\left(1-v^2/c^2\right) = \frac{1}{2\gamma^2} + O(\gamma^{-3})}$$

$$\underline{= i\omega t_0 + i\omega\left[\frac{1}{2}\left(\frac{1}{\gamma^2}+\varphi^2\right)\tau + \frac{c^2}{6R^2}\tau^3 + O(\varepsilon^4)\right]}$$

where $\underline{\varepsilon = O(1/\gamma, \varphi, \tau)}$ is the small parameter

$$\rightarrow \underline{\int dt\, v_y(t)\, e^{i\omega[t-\frac{1}{c}\hat{x}\cdot\vec{y}(t)]}} = \int dt\, v\, \overbrace{\cos\left(\frac{-\pi}{2}+\omega_0\tau\right)}^{=\sin\omega_0\tau} e^{i[\omega t-\frac{\omega}{c}\hat{x}\cdot\vec{y}(t)]}$$

$$\simeq v \int d\tau \left(\omega_0\tau + O(\tau^3)\right) e^{i\omega t_0 + i\omega\frac{1}{2}\left[(1/\gamma^2+\varphi^2)\tau + \frac{c^2}{3R^2}\tau^3 + \dots\right]}$$

$$= \underline{v\omega_0\, e^{i\omega t_0} \int d\tau\, \tau\, e^{i\omega\frac{1}{2}\left[(1/\gamma^2+\varphi^2)\tau + \frac{c^2}{3R^2}\tau^3 + \dots\right]}} =: \underline{\mathcal{F}_\parallel(\omega)}$$

<u>remark</u>: see Jackson p 674 footnote for a justification for ignoring the terms of higher order in τ yet still integrating over all τ. A thorough justification requires tricky asymptotic analysis!

define $\xi := \frac{\omega R}{3c}(1/\gamma^2+\varphi^2)^{3/2}$

$$s = \frac{c\tau}{R(1/\gamma^2+\varphi^2)^{1/2}} \rightarrow ds = \frac{c}{R(1/\gamma^2+\varphi^2)^{1/2}}\, d\tau$$

$$\rightarrow \underline{\mathcal{F}_\parallel(\omega)} = R\omega_0^2\, e^{i\omega t_0}\frac{R^2}{c^2}(1/\gamma^2+\varphi^2)\int_{-\infty}^{\infty} ds\, s\, e^{i\frac{\omega}{2}\left[(1/\gamma^2+\varphi^2)^{3/2}\frac{R}{c}s + \frac{c^2}{3R^2}\frac{R^3}{c^3}(\frac{1}{\gamma^2}+\varphi\right.}$$

$$= \frac{R^3\omega_0^2}{c^2}\, e^{i\omega t_0}(1/\gamma^2+\varphi^2)\int ds\, s\, e^{i\frac{\cancel{\omega}}{\cancel{c}}\left[\frac{3c}{\cancel{\omega R}}\xi\frac{R}{\cancel{c}}s + \frac{R}{3\cancel{c}}\frac{3c}{\cancel{\omega R}}\xi s^3\right]}$$

$$\overset{R\omega_0=v\simeq c}{=} R\, e^{i\omega t_0}(1/\gamma^2+\varphi^2)\int ds\, s\, e^{i\frac{3}{2}\xi\left(s+\frac{1}{3}s^3\right)}$$

$$= Re^{i\omega t_0}(1/\gamma^2+\varphi^2)\int_{-\infty}^{\infty} ds\, s\left[\overset{\rightarrow 0}{\cancel{\cos\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right)\right)}} + i\sin\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right.\right.\right.$$

$$= 2iRe^{i\omega t_0}(1/\gamma^2+\varphi^2)\int_0^\infty ds\, s\sin\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right)\right)$$

$$h \approx \underline{= 2iRe^{i\omega t_0}(1/\gamma^2+\varphi^2)\frac{1}{\sqrt{3}}K_{2/3}(\xi)}$$ with $K_{2/3}$ a modified Bessel fct.

$\mathcal{F}_\parallel(\omega)$ is the spectrum for the parallel polarization direction.

§6.4 → The perpendicular one is given by the vector $\hat{e}_\perp = (-\cos\varphi, 0, \sin$

so the relevant velocity component is

$$\underline{\hat{e}_\perp\cdot\vec{v}(t)} = -v\cos\varphi\sin\omega_0 t = v\cos\left(\frac{\pi}{2}-\varphi\right)\sin\left(-\frac{\pi}{2}+\omega_0\tau\right)$$

$$= -v\sin\varphi\cos\omega_0\tau$$

$$= \underline{-v\varphi + O(\varepsilon^2)}$$

Compare this with

$$\underline{\hat{e}_\parallel\cdot\vec{v}(t)} = v_y(t) = v\cos\left(-\frac{\pi}{2}+\omega_0\tau\right) = v\sin\omega_0\tau = \underline{v\omega_0\tau + O(\varepsilon^2}$$

→ The spectrum $\mathcal{F}_\perp(\omega)$ for the perpendicular polarization is obtained by the substitution $\tau \to \frac{-1}{\omega_0}\varphi$ in the integrand

or $s \to \dfrac{-c}{R(1/\gamma^2+\varphi^2)^{1/2}}\dfrac{1}{\omega_0}\varphi$

$$\to \underline{\mathcal{F}_\perp(\omega)} = -Re^{i\omega t_0}\cancel{(1/\gamma^2+\varphi^2)}\frac{-c}{\cancel{R(1/\gamma^2+\varphi^2)^{1/2}}}\frac{1}{\omega_0}\varphi\int_{-\infty}^{\infty}ds\, e^{i\frac{3}{2}\xi(s+\frac{1}{3}s^3)}$$

$$= -Re^{i\omega t_0}(1/\gamma^2+\varphi^2)^{1/2}\underbrace{\frac{c}{v}}_{\simeq 1}\varphi\int_{-\infty}^{\infty}ds\left[\cos\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right)\right)\right.\left.\cancelto{0}{+i\sin\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right)\right)}\right]$$

$$= -2Re^{i\omega t_0}\varphi(1/\gamma^2+\varphi^2)^{1/2}\int_0^{\infty}ds\cos\left(\frac{3\xi}{2}\left(s+\frac{1}{3}s^3\right)\right)$$

$$= \underline{-2Re^{i\omega t_0}\varphi(1/\gamma^2+\varphi^2)^{1/2}\frac{1}{\sqrt{3}}K_{1/3}(\xi)}$$

For the total spectrum I now have

$$\underline{\underline{\frac{d^2 U}{d\omega\, d\Omega}}} = \left(\frac{d^2 U}{d\omega\, d\Omega}\right)_{\parallel} + \left(\frac{d^2 U}{d\omega\, d\Omega}\right)_{\perp}$$

$$= \frac{\omega^2 e^2}{4\pi^2 c^3}\left[\,|\mathcal{F}_{\parallel}(\omega)|^2 + |\mathcal{F}_{\perp}(\omega)|^2\right]$$

$$= \frac{\omega^2 e^{\cancel{2}}}{\cancel{4}\pi^2 c^3}\,\frac{\cancel{4}R^2}{\cancel{3}}\left[(1/\gamma^2+\varphi^2)^2\,(K_{2/3}(\xi))^2 + \varphi^2(1/\gamma^2+\varphi^2)(K_{1/3}(\xi)\right.$$

$$\underline{= \frac{e^2}{3\pi^2 c}\left(\frac{\omega R}{c}\right)^2 (1/\gamma^2+\varphi^2)^2\left[(K_{2/3}(\xi))^2 + \frac{\varphi^2}{1/\gamma^2+\varphi^2}\,(K_{1/3}(\xi))^2\right.}$$

Jackson (14.83) ✓

with $\underline{\xi} = \underline{\frac{\omega R}{3c}(1/\gamma^2+\varphi^2)^{3/2}}$ a combination of ω, φ, and γ

remark: (1) This is valid for $\gamma \gg 1$, $\varphi \ll 1$

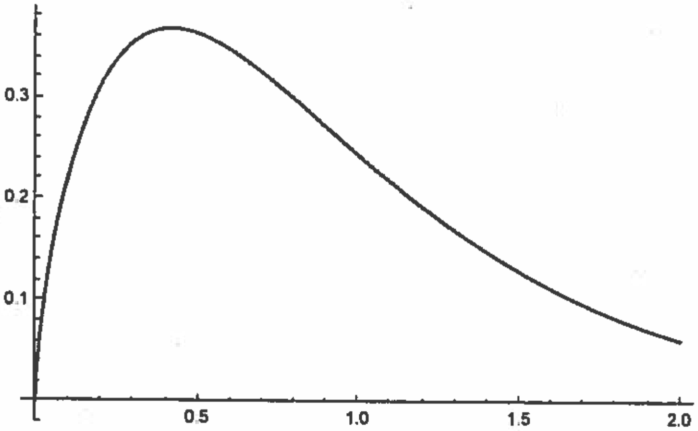
(2) It also involved a short-time approximation $\rightarrow$ It is valid only for $\omega \gg \omega_0$. But we know already that for $\gamma \gg 1$ the spectrum peaks at $\omega \simeq \omega_0 \gamma^3$, so that's okay. This can also be seen from the above result: $\xi \propto \omega\gamma^3$, and both $\xi K_{2/3}(\xi)$ and $\xi K_{1/3}(\xi)$ have a (broad) maximum around $\xi \simeq \underline{1}$.

p4.6-2-11

Now consider the qualitative frequency dependence for fixed small phi, which is dominated by the K[2/3, xi] :

```
In[11]:= Plot[(xi BesselK[2 / 3, xi]) ^2, {xi, 0, 2}]
```

Out[11]=

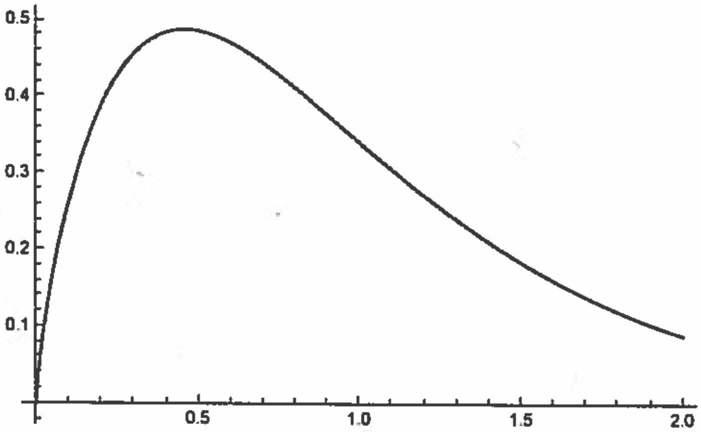


This is indeed qualitatively the same as what we found in part d). The K[1/3, xi] does not change that :

```
In[21]:= gamma = 356
         phi = 1 / gamma
         Plot[
          xi^2 ((BesselK[2 / 3, xi]) ^2 + (phi^2 / (1 / gamma^2 + phi^2)) (BesselK[1 / 3, xi]) ^2),
          {xi, 0, 2}]
Out[21]= 356
```

Out[22]= $\frac{1}{356}$

Out[23]=



Now integrate over the frequency to get the angular dependence.

remark: (1) We know that our approximation is not good at small ω, but since the spectrum peaks at large ω we should still be able to integrate over all ω. Determining how good that is exactly would be hard.

$$\rightarrow \frac{dW}{d\Omega} = \int_0^\infty d\omega \frac{d^2W}{d\omega\, d\Omega} = \frac{3c}{R} \frac{1}{(1/\gamma^2+\varphi^2)^{3/2}} \int_0^\infty d\xi \frac{d^2W}{d\omega\, d\Omega}$$

$$= \frac{3c}{\cancel{R}} \frac{1}{(1/\gamma^2+\varphi^2)^{3/2}} \frac{e^2}{3\pi^2 \cancel{c}} \frac{\cancel{R^2}}{\cancel{c^2}} \left(\frac{3\cancel{c}}{\cancel{R}}\right)^2 \frac{1}{(1/\gamma^2+\varphi^2)^{\cancel{2}}} \cancel{(1/\gamma^2+\varphi^2)^2}$$

$$\times \int_0^\infty d\xi\, \xi^2 \left[(K_{2/3}(\xi))^2 + \frac{\varphi^2}{1/\gamma^2+\varphi^2} (K_{1/3}(\xi))^2 \right]$$

$$= \frac{e^2}{R} \frac{9}{\pi^2} \frac{1}{(1/\gamma^2+\varphi^2)^{5/2}} \left[\underbrace{\int_0^\infty d\xi\, \xi^2 (K_{2/3}(\xi))^2}_{= \frac{7\pi^2}{144}} + \frac{\varphi^2}{1/\gamma^2+\varphi^2} \underbrace{\int_0^\infty d\xi\, \xi^2 (K_{1/3}(\xi))^2}_{= \frac{5\pi^2}{144}} \right]$$

$$= \frac{7}{16} \frac{e^2}{R} \frac{1}{(1/\gamma^2+\varphi^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\varphi^2}{1/\gamma^2+\varphi^2} \right]$$

Jackson (14.84) ✓ see also Tomboulian & Hartman, Phys Rev 102, 1423 (1955).

This confirms the qualitative argument from § 6.3: The bulk of the radiation is emitted into a narrow range of angles $|\varphi| \lesssim 1/\gamma$ about the orbital plane.

4.7.1.) a) The sphere is homogeneous and isotropic, so the polarization vector is proportional to the external field:

$$\vec{P} \propto \vec{E}_{ext}$$

The dipole moment is the volume times $\vec{P}$ ->

① $$\vec{d} \propto a^3 \vec{E}_{ext}$$

The proportionality factor will depend on the geometry and on the material properties, i.e., on ε:

$$\rightsquigarrow \quad \underline{\underline{\vec{d} = f(\varepsilon) a^3 \vec{E}_{ext}}}$$

Now, for $\varepsilon \to 1$ the susceptibility and hence $\vec{P}$ and $\vec{d}$ vanish, so we must have

① $$\underline{f(\varepsilon \to 1) = 0}$$

For $\varepsilon \to \infty$ the susceptibility diverges

-> There is no field inside the sphere

-> $\vec{P} = \vec{E}_{ext} \times$ (a purely geometric factor)

$$\rightsquigarrow \quad \underline{\underline{f(\varepsilon \to \infty) = \text{const}}}$$

① with the constant entirely determined by the geometry

b) §3.4 Woolley -> For electric dipole radiation, the total radiation power is $\mathcal{P}_{ed} = \frac{2}{3c^3}(\ddot{\vec{d}})^2 = \frac{2\omega^4}{3c^3}\vec{d}^2$

and from part a) we have

① $$\vec{d} = f(\varepsilon) a^3 \vec{E}_{ext}$$

p4.7.1-2

§3.4 ⇝ For the electric dipole to give the dominant contribution, $\vec{E}_{ext}$ must vary slowly over the sphere, i.e., $\omega = ck \ll c/a$

① or $\underline{\lambda = 2\pi/k \gg a}$

In this limit, we have

$$\mathcal{P}_{scatt} \approx \mathcal{P}_{ed} = \frac{2\omega^4}{3c^3}\vec{d}^2 = \frac{2\omega^4}{3c^3}\left(f(\epsilon)a^3\right)^2 \vec{E}_{ext}^2$$

§7.2 ⇝ The Poynting vector is $|\vec{P}| = \frac{c}{4\pi}\vec{E}_{ext}^2$

and the scattering cross section is

$$\underline{\underline{\sigma}} = \frac{\mathcal{P}_{scatt}}{|\vec{P}|} = \frac{8\pi}{3}\frac{\omega^4}{c^4}a^6\left(f(\epsilon)\right)^2$$

$$= \underline{\underline{\frac{8\pi}{3}(ka)^4\left(f(\epsilon)\right)^2 a^2}}$$

① with $\underline{k = \omega/c = 2\pi/\lambda}$ the wave number of the radiation