

0.2.1 The brachistochrone problem

A point mass glides without friction on an inclined plane (inclination angle α) from point P_1 to point P_2 on a path \mathcal{C} according to Galilean mechanics.

- Use energy conservation to find the velocity as a function of y , using the coordinate system in the sketch.
- Write the passage time from P_1 to P_2 in the form

$$T = \int_{x_1}^{x_2} dx L(y, y')$$

with y considered a function of x and $y' = dy/dx$, and determine the Lagrangian L . Use the fact that Jacobi's integral is constant to find an ODE for y .

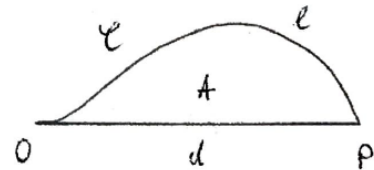
- Substitute $y' = \text{ctg } t$ to write the brachistochrone in a parametric form, $y = y(t)$, $x = x(t)$. Finally, reparameterize by defining $\tau = 2t - \pi$ to cast the result in the standard parametric form for a cycloid.
- Express the passage time as a function of the terminal y -velocity, $y'_2 = (dy/dx)_{P_2}$.
- Find the passage time for the shortest path from P_1 to P_2 (as opposed to the brachistochrone) as a function of the same y'_2 .
- Discuss the ratio of the two passage times as a function of y'_2 .

hint: The parameter value t_2 for the brachistochrone at the end point P_2 is a known function of y'_2 . It therefore suffices to discuss the passage time as a function of t_2 .

(18 points)

0.2.2 Dido's problem

An area A in the x - y -plane is enclosed by straight line between two points O and P that are a distance d apart, and a path \mathcal{C} with end points O and P and length $\ell > d$. Find the path \mathcal{C} that maximizes A .



(6 points)

0.2.3 Geodesics on the 2-sphere

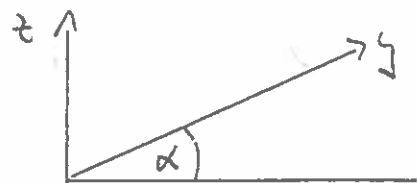
Show that the geodesics on the 2-sphere are great circles.

hint: There are various ways of doing this. One is to set up the problem of geodesics in \mathbb{R}_3 with the constraint that the desired paths $\vec{x}(t)$ must lie on the sphere. Now use the Euler-Lagrange equations for the constrained problem to show that $\vec{\ell} = \vec{x} \times \dot{\vec{x}} = \text{const}$, where $\vec{p} = \partial L / \partial \dot{\vec{x}}$, with L the appropriate Lagrangian.

(5 points)

0.2.1.) constraint: $z = y \sin \alpha$

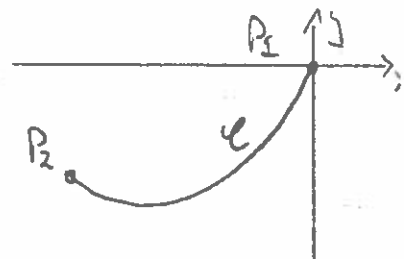
let $P_1 = (0, 0, 0)$ in \mathcal{L}



a) Energy conservation \rightarrow

$$\frac{m}{2} v^2 = -mgt = -mgy \sin \alpha$$

① \rightarrow $v(x, y) = v(y) = \sqrt{2gy \sin \alpha} =: \sqrt{-c_1 y}$ when $c_1 = 2g \sin \alpha$



b) length of infinitesimal path element: $dl = dt \sqrt{\dot{x}^2 + \dot{y}^2}$

Time needed to move across dl : $dT = dl/v(y)$

\rightarrow Passage time

① $\underline{T} = \int_{t_1}^{t_2} dt \frac{1}{v(y)} \sqrt{\dot{x}^2 + \dot{y}^2} = \int_{x_1}^{x_2} dx \frac{1}{v(y)} \sqrt{1 + y'^2}$

$= \int_{x_1}^{x_2} dx L(y, y')$ with $L(y, y') = \frac{1}{v(y)} \sqrt{1 + y'^2}$

NO p.2.3 remark (2) (iii) \rightarrow Jacobi's integral is a constant of motion, i.e.,

$$H(y, y') = y' \frac{\partial L}{\partial y'} - L = \frac{y'^2}{v(y) \sqrt{1 + y'^2}} - \frac{1}{v(y) \sqrt{1 + y'^2}} = \frac{-1}{v(y) \sqrt{1 + y'^2}} = \text{const}$$

① \rightarrow $y(1 + y'^2) = \text{const} =: c_2 < 0$

c) hbslhl $y' = \cot t$, $t = \arccot y'$

$\rightarrow y = \frac{c_1}{1+y'^2} = \frac{c_1}{1+\cot^2 t} = c_1 \sin^2 t = \frac{1}{2} c_1 (1 - \cos 2t)$

$dx = \frac{dy}{y'} = \frac{2c_1 \sin t \cos t dt}{\cot t} = 2c_1 \sin^2 t dt = c_1 (1 - \cos 2t) dt$

$\rightarrow x = c_2 + c_1 t - \frac{1}{2} c_1 \sin 2t = \frac{1}{2} c_1 (2t - \sin 2t) + c_2$

initial conditions: $y_1 = y(t=0) = 0 \checkmark$

$x_1 = x(t=0) = c_2 \stackrel{!}{=} 0 \rightarrow c_2 = 0$

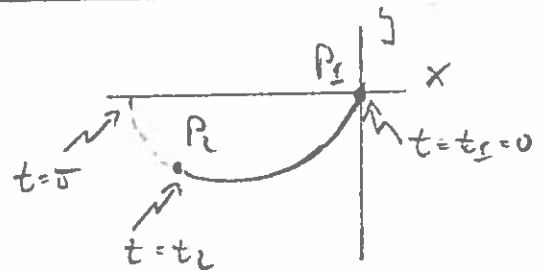
let $c = \frac{1}{2} c_1 \rightarrow$

$$\begin{cases} x(t) = c(2t - \sin 2t) \\ y(t) = c(1 - \cos 2t) \end{cases}$$

Directrix is
in parametric form

remark: (1) This is for $x_2 < 0$

If $x_2 > 0$, we'd do
 $t < 0$



(2) c is a scale factor that must be chosen such that P_2 lies on the curve.

To bring this in standard form, let $\tilde{t} = 2t - \delta$, $\tilde{x} = -x + \delta c$

$\rightarrow \tilde{x}(\tilde{t}) = -c(\tilde{t} + \delta - \sin(\tilde{t} + \delta)) + \delta c = -c(\tilde{t} + \sin \tilde{t})$
 $\underline{y(\tilde{t})} = c(1 - \cos(\tilde{t} + \delta)) = c(1 + \cos \tilde{t}) = -c(-1 - \cos \tilde{t})$ } cycloid

12/1) a) Polstelle \rightarrow

$$\begin{array}{|l} x(t) = c(2t - \omega t) \\ y(t) = c(1 - \omega t) \end{array}$$

$$c < 0$$

$$t = \arccos \frac{y'}{c}$$

$$y' = dy/dx$$

$$\rightarrow \dot{x}(t) = 2c(1 - \omega t)$$

$$\dot{y}(t) = 2c\omega t$$

$$\begin{aligned} \rightarrow \dot{x}^2 + \dot{y}^2 &= 4c^2(1 - \omega t)^2 + 4c^2\omega^2 t^2 = 4c^2 - 8c^2\omega t + 4c^2 + 8c^2\omega t - 4c^2\omega^2 t^2 \\ &= 16c^2\omega^2 t^2 \end{aligned}$$

①

\rightarrow passage time for the brechinstrom

$$\underline{T_b} = \int_0^{t_2} dt \frac{1}{|v_y|} (-4c)\omega t = \frac{-4c}{|v_y|} \int_0^{t_2} dt \frac{1}{|1 - \omega t|} \omega t$$

$$= \frac{-4c}{|v_y|} \int_0^{t_2} dt \frac{\omega t}{|1 - \omega t|} = \frac{-4c}{|v_y|} \int_0^{t_2} dt \frac{1}{|1 - \omega t|} \omega t$$

$$= 2\sqrt{-2c/a} t_2 \quad \text{with} \quad t_2 = \arccos \frac{y_2}{c}$$

①

e) straight line: $x(\tau) = x_2 \tau$ $v_1 = 0, v_2 = 1$ ($y_2 < 0$)
 $y(\tau) = y_2 \tau$

\rightarrow passage time for straight line

$$\underline{T_s} = \int_0^1 d\tau \frac{1}{|v_y|} \sqrt{x_2^2 + y_2^2} = \sqrt{x_2^2 + y_2^2} \frac{1}{|v_y|} \int_0^1 d\tau \tau^{-1/2} = \frac{2}{|v_y|} \sqrt{x_2^2 + y_2^2}$$

①

But we know that the point (x_2, y_2) lies on the brechinstrom:

c) $\rightarrow x_2 = c(2t_2 - \omega t_2), y_2 = c(1 - \omega t_2)$

$$\begin{aligned} \rightarrow \underline{x_2^2 + y_2^2} &= c^2(4t_2^2 - 4t_2\omega t_2 + \omega^2 t_2^2 + 1 - 2\omega t_2 + \omega^2 t_2^2) \\ &= c^2(2(1 - \omega t_2) + 4t_2^2 - 4t_2\omega t_2) \\ &= c^2(4\omega^2 t_2^2 + 4t_2^2 - 4t_2\omega t_2) \end{aligned}$$

①

$$p=0: 2.1-4$$

$$= 4c^2 (\dot{w}^2 t_2 + \dot{t}_2^2 + t_2 \dot{w} \dot{t}_2)$$

$$\begin{aligned} \rightarrow \underline{\underline{T_s}} &= \frac{\lambda}{\sqrt{-oc}} \frac{1}{\sqrt{1-w\dot{t}_2}} (-\sqrt{2c}) \sqrt{\dot{w}^2 t_2 + \dot{t}_2^2 + t_2 \dot{w} \dot{t}_2} \\ &= \frac{+\cancel{2}\cancel{\lambda}c}{\sqrt{-oc}} \frac{1}{\sqrt{2} \dot{w} t_2} \sqrt{\dot{w}^2 t_2 + \dot{t}_2^2 + t_2 \dot{w} \dot{t}_2} \\ &= \sqrt{-2c/a} \frac{+2}{\dot{w} t_2} \sqrt{t_2^2 + \dot{w}^2 t_2 + t_2 \dot{w} \dot{t}_2} \quad \text{with } t_2 \text{ fixed} \end{aligned}$$

f) $t_2 \rightarrow 0$ ($\rightarrow y_2' \rightarrow \infty$)

$$\begin{aligned} \underline{\underline{T_s(t_2 \rightarrow 0)}} &= \sqrt{-2c/a} \frac{-2}{t_2(1+O(t_2^2))} \sqrt{t_2^2 + \dot{t}_2^2 - \frac{2}{6} t_2^4 - 2\dot{t}_2^2 + \frac{8}{6} t_2^4 + O(t_2^6)} \\ &= -2 \sqrt{-2c/a} t_2 [1+O(t_2^2)] \end{aligned}$$

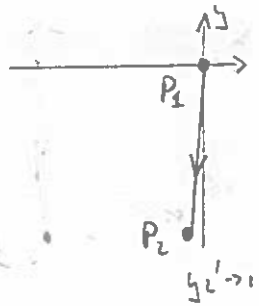
$$= \underline{\underline{T_b [1+O(t_2^2)]}}$$

This is the con when the end point

P_2 has done to the full line.

$\rightarrow y_2' \rightarrow \infty, t_2 \rightarrow 0$, ed brachistochron

ed straight line are almost identical.

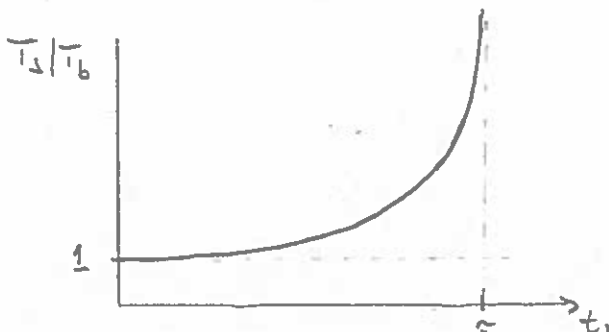
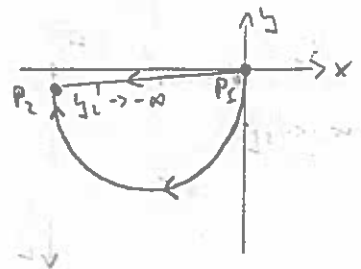


$$\underline{\underline{t_2 = -\pi - \epsilon}} \quad (\rightarrow y_2' \rightarrow -\infty)$$

$$\underline{\underline{T_s(t_2 = -\pi - \epsilon)}} = \sqrt{-2c/a} \frac{2\pi}{\epsilon} [1+O(\epsilon^2)] = \underline{\underline{T_b(t_2 = -\pi) \frac{1}{\epsilon} [1+O(\epsilon^2)]}}$$

This is the con when P_2 has done to

the x-axis $\rightarrow y_2' \rightarrow -\infty$, ed the straight-line path is very slow.



0.2.2.) d.o.f 2.4 \times (5) $\rightarrow A = \frac{1}{2} \oint dt [x(t) \dot{y}(t) - y(t) \dot{x}(t)]$

$$L = \oint dt \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

① d.o.f 2.4 then $\rightarrow L = \frac{1}{2}(x\dot{y} - y\dot{x}) + \frac{1}{2}\lambda \sqrt{\dot{x}^2 + \dot{y}^2}$

EL eq: $\frac{1}{2}\dot{y} = \frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -\frac{1}{2}\dot{y} + \frac{1}{2}\lambda \frac{d}{dt} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$

$$\rightarrow \underbrace{y - y_0 = \lambda \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}_{(1)}$$

$$-\frac{1}{2}\dot{x} = \frac{\partial L}{\partial y} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{1}{2}\dot{x} + \frac{1}{2}\lambda \frac{d}{dt} \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\rightarrow \underbrace{x - x_0 = -\lambda \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}_{(2)}$$

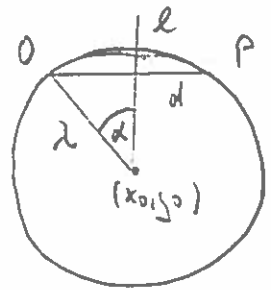
$$\rightarrow \boxed{(x - x_0)^2 + (y - y_0)^2 = \lambda^2}$$

circle with center (x_0, y_0) , radius λ

Now $\overline{OP} = d = 2\lambda \sin \alpha$

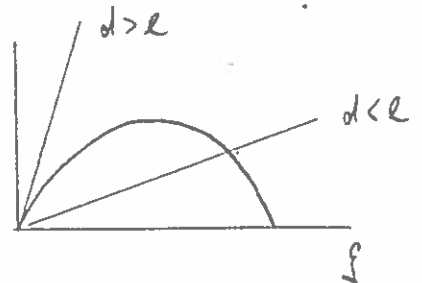
and $l = 2\lambda \alpha$

$$\rightarrow \boxed{\frac{d}{2\lambda} = \sin \frac{l}{2\lambda}}$$
 determines λ



Define $f := l/2\lambda$

$$\rightarrow \boxed{\sin f = \frac{d}{2\lambda} f}$$



Graphic solution \rightarrow

1st con: $\underline{d \geq l}$ no solution

$\underline{d < l}$ exactly one solution, valid obviously is a maximum

p-0.2.3

0.2.3.) Consider a path in \mathbb{R}^3 : $\vec{x}(t) = (x_1(t), x_2(t), x_3(t))$

length of curve:
$$l = \int_{t_-}^{t_+} dt \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t) + \dot{x}_3^2(t)}$$

$$=: \int_{t_-}^{t_+} L_1(\dot{\vec{x}})$$

①

Boundary condition: $x_1^2(t) + x_2^2(t) + x_3^2(t) = 1 \quad (+)$

or
$$0 = \int_{t_-}^{t_+} dt (x_1^2(t) + x_2^2(t) + x_3^2(t) - 1)$$

$$=: \int_{t_-}^{t_+} L_2(\vec{x})$$

①

\Rightarrow consider $L = L_1 - \lambda L_2$

$$L(\vec{x}, \dot{\vec{x}}) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} - \lambda (x_1^2 + x_2^2 + x_3^2 - 1)$$

Euler-Lagrange $\rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$

$$\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}} = -2\lambda x_i \quad (*)$$

①

Now consider $\vec{l} := \vec{x} \times \vec{p}$ with $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$

$\rightarrow \vec{p} \parallel \dot{\vec{x}}$, or $\vec{p} \times \dot{\vec{x}} = 0$

Further, $(*) \rightarrow \dot{\vec{p}} = -2\lambda \dot{\vec{x}} \rightarrow \vec{p} \times \dot{\vec{x}} = 0$

$\rightarrow \dot{\vec{l}} = \dot{\vec{x}} \times \vec{p} + \vec{x} \times \dot{\vec{p}} = 0 \rightarrow \vec{l} = \text{const} \quad (**)$

①

and $\vec{l} \cdot \dot{\vec{x}} = \dot{\vec{x}} \cdot (\vec{x} \times \vec{p}) = 0 \rightarrow \boxed{l_1 x_1 + l_2 x_2 + l_3 x_3 = 0}$ will $l_{1,2,3} = \text{const}$

①

(**) describes a plane that contains the origin } \rightarrow The geodesics
 (+) describes a sphere whose center is the origin