PHYS 622 Classical Electrodynamics W 2021

Problem Assignment # 2 01/15/2021
due 01/22/2021

0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)] \right)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

   i) $F = \int dx\, \varphi(x)$

   ii) $F = \int dx\, \varphi^2(x)$

   iii) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
   hint: Integrate by parts and assume that the boundary terms vanish.

   iv) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

   remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

0.2.5. **Massive scalar field**

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\, (\partial_\mu\phi(x))\, (\partial^\mu\phi^*(x)) - \frac{m^2}{2}\, |\phi(x)|^2$$

with ϕ^* the complex conjugate of ϕ. What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(2 points)

. . . **/over**

0.2.6. **Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields**

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

   hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\, \boldsymbol{E} \times \boldsymbol{B} / \boldsymbol{B}^2$$

   Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.2.7. **Harmonic oscillator coupled to a magnetic field**

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0, charge e). Put the oscillator in a homogeneous time-independent magnetic field $\boldsymbol{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $\boldsymbol{B}$.

(4 points)

0.2.4.) a) i) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left(\cancel{\varphi(y)} + \varepsilon\,\delta(y-x) - \cancel{\varphi(y)} \right) = \int dy\, \delta(y-x) = \underline{\underline{1}}$

ii) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[(\varphi(y) + \varepsilon\,\delta(y-x))^2 - \varphi^2(y) \right]$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[2\varepsilon\,\varphi(y)\,\delta(y-x) + O(\varepsilon^2) \right] = \underline{\underline{2\varphi(x)}}$

1

(iii) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[\left(\varphi'(y) + \varepsilon \frac{d}{dy}\delta(y-x)\right)^2 - (\varphi'(y))^2 \right]$

$= 2\int dy\, \varphi'(y) \frac{d}{dy}\delta(y-x) = \underline{\underline{-2\varphi''(x)}}$

(iv) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[V\left(\varphi'(y) + \varepsilon \frac{d}{dy}\delta(y-x)\right) - V(\varphi'(y)) \right]$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[\varepsilon\, V'(\varphi'(y)) \frac{d}{dy}\delta(y-x) + O(\varepsilon^2) \right]$

$= \underline{\underline{-V''(\varphi'(x))\,\varphi''(x)}}$

1

b) $\underline{\underline{0}} \overset{!}{=} \frac{\delta}{\delta\varphi(x)} \int d^4y\, \mathcal{L}(\varphi(y), \partial_\mu\varphi(y))$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int d^4y \left[\mathcal{L}(\varphi(y) + \varepsilon\,\delta(y-x),\, \partial_\mu\varphi(y) + \varepsilon\,\partial_\mu\delta(y-x)) - \mathcal{L}(\varphi(y), \partial_\mu\varphi(y)) \right]$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int d^4y \left[\varepsilon\,\delta(y-x) \frac{\partial\mathcal{L}}{\partial\varphi(y)} + \varepsilon(\partial_\mu\delta(y-x)) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(y))} + O(\varepsilon^2) \right]$

$= \underline{\underline{\frac{\partial\mathcal{L}}{\partial\varphi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))}}}$ ✓

1

0.2.5.) a) Treat $\phi(x)$ and $\phi^*(x)$ as independent fields.

Minimizing with respect to ϕ^* yields

$$\underline{(\partial_\mu \partial^\mu + m^2)\,\phi(x) = 0}$$

and minimizing with respect to ϕ just yields the c.c.:

$$\underline{(\partial_\mu \partial^\mu + m^2)\,\phi^*(x) = 0}$$

1

c) Under $\phi(x) \to \phi(x)\, e^{i\Delta(x)}$ we have

$$|\phi(x)|^2 \to |\phi(x)|^2$$

and

$$\partial_\mu \phi(x) \to (\partial_\mu \phi(x))\, e^{i\Delta(x)} + i(\partial_\mu \Delta(x))\,\phi(x)\, e^{i\Delta(x)}$$

$$\partial^\mu \phi^*(x) \to (\partial^\mu \phi^*(x))\, e^{-i\Delta(x)} - i(\partial^\mu \Delta(x))\,\phi^*(x)\, e^{-i\Delta(x)}$$

$$\to \underline{\partial_\mu \phi(x)\,\partial^\mu \phi^*(x)} \to \partial_\mu \phi(x)\,\partial^\mu \phi^*(x) - i(\partial^\mu \Delta(x))(\partial_\mu \phi(x))\,\phi^*(x)$$
$$+ i(\partial_\mu \Delta(x))(\partial^\mu \phi^*(x))\,\phi(x)$$
$$+ (\partial_\mu \Delta(x))(\partial^\mu \Delta(x))\,|\phi(x)|^2$$

$$\underline{\neq \partial_\mu \phi\, \partial^\mu \phi^*}$$

$\to$ The Lagrangian is not invariant

1

0.2.6.) a) Eq. of motion: $m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

Let $\vec{B} = (0,0,B)$ and $\vec{E} = (0,E_y,E_z)$

$$\rightarrow \boxed{\begin{array}{l} m\ddot{x} = \frac{e}{c}\dot{y}B \quad (1) \\ m\ddot{y} = eE_y - \frac{e}{c}\dot{x}B \quad (2) \\ m\ddot{z} = eE_z \quad (3) \end{array}}$$

(3) $\rightarrow$ $\underline{\underline{z(t) = z_0 + v_z^0 t + \frac{eE_z}{2m}t^2}}$

1

b) Define $\xi := x + iy$

(1) + i·(2) $\rightarrow$ $m\ddot{\xi} = ieE_y - i\frac{eB}{c}\dot{\xi}$

Define $\omega := \frac{eB}{mc}$ cyclotron frequency

$$\rightarrow \boxed{\ddot{\xi} + i\omega\dot{\xi} = i\frac{e}{m}E_y} \quad (*)$$

Special solution of inhomogeneous eq.: $\dot{\xi} = \frac{eE_y}{m\omega}$

General solution of homogeneous eq.: $\dot{\xi} = a e^{-i\omega t}$ $(a \in \mathbb{C})$

1 $\rightarrow$ $\underline{\underline{\dot{\xi}(t) = a e^{-i\omega t} + eE_y/m\omega}}$ is the most general solution of (*).

c) With $a = be^{i\alpha}$, $b,\alpha \in \mathbb{R}$

$\rightarrow \dot{\xi} = b e^{-i(\omega-\alpha)t} + eE_y/(m\omega)$

$\rightarrow$ α just shifts the zero of time $\rightarrow$ $\underline{\underline{\alpha = 0 \text{ w.l.o.g.}}}$

$\rightsquigarrow \quad \dot{x} + i\dot{y} = b\cos\omega t - ib\sin\omega t + eE_y/m\omega$

$\rightsquigarrow \quad \boxed{\begin{array}{l} \dot{x} = b\cos\omega t + eE_y/m\omega \\ \hline \dot{y} = -b\sin\omega t \end{array}} \quad (**)$

(1)

$\rightsquigarrow \quad \underline{\langle \dot{y} \rangle = 0}\,, \quad \underline{\langle \dot{x} \rangle} = eE_y/m\omega = \underline{cE_y/B}$ time-averaged velocity

$= \dfrac{cE_y B}{B^2} = \underline{c(\vec{E}\times\vec{B})_x/B^2}$

in general:

$\underline{\underline{\langle \vec{v} \rangle = \frac{c}{B^2}\vec{E}\times\vec{B}}}$ drift velocity

condition for $v \ll c$: $\quad \underline{\underline{E_y/B \ll 1}}$ necessary and sufficient condition for non-relativistic approximation

(1)

d) Choose $x(t=0) = 0 = y(t=0)$ w.l.o.g.

$(**) \rightsquigarrow \quad \boxed{\begin{array}{l} x(t) = \frac{b}{\omega}\sin\omega t + \frac{cE_y}{B}t \\ \hline y(t) = \frac{b}{\omega}(\cos\omega t - 1) \end{array}}$

$\rightsquigarrow$ The path is a trochoid.

To visualize it, put $\underline{\omega = 1}$ and define $C = cE_y/B$.

$\rightsquigarrow \quad \boxed{\begin{array}{l} x(t) = b\sin t + Ct \\ \hline y(t) = b(\cos t - 1) \end{array}}$ This is the projection of the path onto the x-y plane

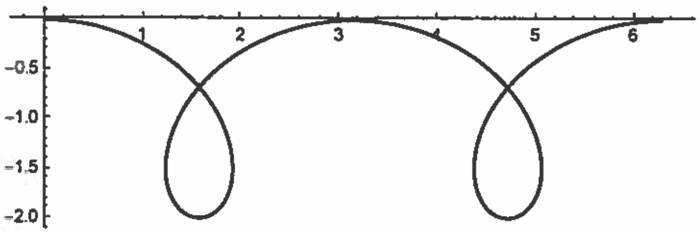
(1)

p10.26-7

For C < b the trochoid has loops :

In[21]:=
```
b = 1;
c = 0.5;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}]
```

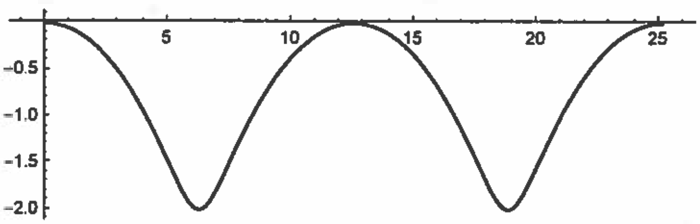
Out[25]=



For C > b it does not :

In[86]:=
```
b = 1;
c = 2;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

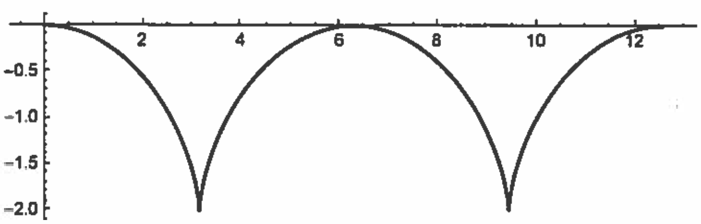
Out[90]=



And for C = b it degenerates into a cycloid :

In[91]:=
```
b = 1;
c = 1;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[95]=



1

0.2.7.) In addition to the restoring force $-m\omega_0^2 x$, the particle is subject to a Lorentz force

$$\frac{e}{c}\vec{v}\times\vec{B} = \frac{e}{c}(\dot{y}B, -\dot{x}B, 0)$$

→ The eqs of motion are

$$\boxed{\begin{aligned} \ddot{x} + \omega_0^2 x &= \Omega\dot{y} \quad (1)\\ \ddot{y} + \omega_0^2 y &= -\Omega\dot{x} \quad (2)\\ \ddot{z} + \omega_0^2 z &= 0 \quad (3)\end{aligned}}$$

(1) with $\Omega := eB/mc$ the cyclotron frequency.

(3) → For oscillations in the $\underline{z\text{-direction}}$, the frequency

(1) $\underline{\underline{\omega = \omega_0}}$ is unchanged

Define $\xi := x + iy$ and consider

(1) $+ i\cdot$(2) → $\boxed{\ddot{\xi} + \omega_0^2\xi = -i\Omega\dot{\xi}}$

(1) ansatz: $\xi(t) = \xi_0 e^{i\omega t}$

$$\to \quad -\omega^2 + \omega_0^2 = \omega\Omega$$

or $\omega^2 + \Omega\omega - \omega_0^2 = 0$

$$\to \quad \omega = \frac{1}{2}\left(-\Omega \pm \sqrt{\Omega^2 + 4\omega_0^2}\right) = \pm\sqrt{\omega_0^2 + \Omega^2/4} - \Omega/2$$

→ The motion in the x-y plane is oscillatory, and the eigenfrequencies are

(1) $$\underline{\underline{\omega_\pm = \sqrt{\omega_0^2 + \Omega^2/4} \pm \Omega/2}}$$