

## Problem Assignment # 3

01/22/2021  
due 01/29/2021

## 0.2.8. Relativistic motion in parallel electric and magnetic fields

Consider a relativistic charged particle (mass  $m$ , charge  $e$ ) in parallel homogeneous electric and magnetic fields  $\mathbf{E} = (0, 0, E)$ ,  $\mathbf{B} = (0, 0, B)$ .

- Show that the equation of motion for the  $z$ -component of the momentum  $p_z$  decouples from  $p_x$  and  $p_y$ , and that the momentum perpendicular to the  $z$ -axis is a constant of motion:  $p_x^2 + p_y^2 \equiv p_\perp^2 = \text{const.}$
- Choose the zero of time such that  $p_z(t = 0) = 0$ , and show that with a suitable chosen origin the  $z$ -component of the particle's position can be written

$$z(t) = \frac{1}{eE} \sqrt{T_0^2 + c^2 e^2 E^2 t^2}$$

where  $T_0$  is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time  $t = 0$ .

*hint:* Recall Einstein's law of falling bodies, ch. 0 §3.3.

- Introduce a parameter  $\varphi$  via  $d\varphi/dt = ceB/T(t)$ , with  $T(t)$  the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$x = \frac{cp_\perp}{eB} \sin \varphi \quad , \quad y = \frac{cp_\perp}{eB} \cos \varphi \quad , \quad z = \frac{T_0}{eE} \cosh(E\varphi/B)$$

and explicitly find the relation between  $\varphi$  and  $t$ .

*hint:* Consider  $\pi := p_x + ip_y$  and note that  $|\pi| = p_\perp = \text{const.}$  by the result of part a).

- Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)

## 1.1.1. Dual field tensor

Show that the dual field tensor  $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$  obeys  $\partial_\mu \tilde{F}^{\mu\nu}(x) = 0$ .

*hint:* First show that  $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$ , and then relate  $\partial_\mu \tilde{F}^{\mu\nu}(x)$  to that expression.

(2 points)

... /over

### 1.1.2. Ginzburg-Landau theory

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\text{GL}} = \int d\mathbf{x} \left[ r |\phi(\mathbf{x})|^2 + c |[\nabla - iq\mathbf{A}(\mathbf{x})]\phi(\mathbf{x})|^2 + u |\phi(\mathbf{x})|^4 + \frac{1}{16\pi\mu} F_{ij}(\mathbf{x}) F^{ij}(\mathbf{x}) \right]$$

Here  $\mathbf{x} \in \mathbb{R}^3$ , and  $\phi(\mathbf{x})$  is a complex-valued field that describes the superconducting matter,  $\mathbf{A}$  is the Euclidian vector field that comprises the spatial components of the 4-vector  $A^\mu = (A^0, \mathbf{A})$ , and  $F_{ij} = \partial_i A_j - \partial_j A_i$  ( $i, j = 1, 2, 3$ ).  $\mu$  and  $q$  are coupling constants that characterize the vector potential and its coupling to the matter, and  $r$ ,  $c$  and  $u$  are further parameters of the theory.

- a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of  $S_{\text{GL}}$  with respect to all independent fields. (See Problem 0.2.4. You may want to double check against what you get from the Landau-Lifshitz method we used in class.)
- b) Show that this theory is invariant under gauge transformations  $\phi(x) \rightarrow \phi(\mathbf{x}) e^{iq\lambda(\mathbf{x})}$ ,  $\mathbf{A}(\mathbf{x}) \rightarrow \mathbf{A}(\mathbf{x}) + \nabla\lambda(\mathbf{x})$ .
- c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 0.2.5, can be made gauge invariant by coupling  $\phi(x)$  to the electromagnetic vector potential  $A^\mu(x)$ .

*hint:* Replace the 4-gradient  $\partial_\mu$  by  $D_\mu = \partial_\mu - iqA_\mu$  and add the Maxwell Lagrangian.

*note:* If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)

0.2.8.) a) The Lagrangian is

$$L = L_0 + eEz + \frac{e}{c} \vec{v} \cdot \vec{A}$$

with  $L_0 = -mc^2 \sqrt{1 - v^2/c^2}$

and  $\vec{A} = \frac{1}{c} \begin{pmatrix} -\vec{v}_y \\ \vec{v}_x \\ 0 \end{pmatrix} \rightarrow \vec{v} \times \vec{A} = \begin{pmatrix} 0 \\ 0 \\ v_x v_y \end{pmatrix} \checkmark$

$\rightarrow \partial L / \partial z = eE$  independent of  $\vec{v}$

$\rightarrow \dot{p}_z = eE$  conserved for  $p_x, p_y$

$\rightarrow p_z(t) = eEt$  (with  $t=0$  done and  $u_{0z}$   $p_z(t=0)=0$ )

The forces in  $x$  and  $y$ -direction are

$$F_x = \frac{e}{c} (\vec{v} \times \vec{A})_x = \frac{e v_y}{c}, \quad F_y = \frac{e}{c} (\vec{v} \times \vec{A})_y = -\frac{e v_x}{c}$$

$\rightarrow \dot{p}_x = \frac{e v_y}{c}, \quad \dot{p}_y = -\frac{e v_x}{c} \quad (*)$

where  $\vec{p} = (p_x, p_y, p_z) = \frac{\partial L}{\partial \vec{v}} = \frac{m \vec{v}}{\sqrt{1 - v^2/c^2}}$  is the momentum

$\rightarrow \frac{d}{dt} (p_x^2 + p_y^2) = 2(p_x \dot{p}_x + p_y \dot{p}_y) = \frac{2meE}{c} \frac{1}{\sqrt{1 - v^2/c^2}} (v_x v_y - v_y v_x) = 0$

$\rightarrow \underline{p_x^2 + p_y^2 =: p_\perp^2 = \text{const}}$

b) The kinetic energy is

$$T = \vec{v} \frac{\partial L_0}{\partial \vec{v}} - L_0 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \sqrt{m^2 c^4 + c^2 p^2}$$

$$= \sqrt{m^2 c^4 + c^2 p_\perp^2 + c^2 p_z^2} = \sqrt{T_0^2 + c^2 e^2 E^2 t^2}$$

where  $T_0 = \sqrt{m^2 c^4 + c^2 p_\perp^2} = T(t=0)$

$$\text{mit } \vec{v} = \frac{\vec{p}}{m' \gamma c} = \frac{c \vec{p}}{m' c^2 \gamma} = \frac{c \vec{p}}{T} \quad (**)$$

$$\rightarrow \dot{z} = \frac{c^2 p_z}{T} = \frac{c^2 e E t}{\sqrt{T_0^2 + c^2 e^2 E^2 t^2}}$$

$$\rightarrow \underline{z(t)} = z_0 + c^2 e E \int_0^t \frac{\tau}{\sqrt{T_0^2 + c^2 e^2 E^2 \tau^2}} d\tau = z_0 + \frac{c^2 e E}{c^2 e^2 E^2} \frac{1}{2} \int_0^{c e E t} \frac{dx}{\sqrt{T_0^2 + x^2}}$$

$$= z_0 + \frac{1}{c E} \frac{1}{2} \int_0^{c e E t} \frac{dx}{1+x^2} = z_0 + \frac{T_0}{c E} \sqrt{1 + c^2 e^2 E^2 t^2 / T_0^2} - \frac{T_0}{c E}$$

$$= \frac{1}{c E} \sqrt{T_0^2 + c^2 e^2 E^2 t^2} \quad \text{mit } z_0 = T_0 / c E$$

cf PHYS 611 Problem 21 ✓

c) Define  $\tilde{\pi} = p_x + i p_y$

$$(b) \rightarrow \dot{\tilde{\pi}} = \frac{e \vec{v}}{c} \cdot (-i) (v_x + i v_y) = \frac{e \vec{v}}{c} \cdot (-i) \frac{c^2}{T} \tilde{\pi} = -i \frac{e \vec{v}}{T} \tilde{\pi}$$

Define  $\varphi$  by  $\frac{e \vec{v}}{T} dt = d\varphi$

$$\rightarrow \frac{d\tilde{\pi}}{d\varphi} = -i \tilde{\pi} \quad \rightarrow \underline{\tilde{\pi} = p_{\perp} e^{-i\varphi}}$$

$$\rightarrow \underline{\vec{v}} = p_{\perp} e^{-i\varphi} \frac{1}{c^2} (v_x + i v_y) = \frac{1}{c^2} \frac{d}{dt} (x + i y) = \frac{T}{c^2} \frac{e \vec{v}}{T} \frac{d}{d\varphi} (x + i y)$$

$$= \frac{e \vec{v}}{c} \frac{d}{d\varphi} (x + i y)$$

$$\rightarrow \frac{dx}{d\varphi} = \frac{c p_{\perp}}{e T} \cos \varphi, \quad \frac{dy}{d\varphi} = -\frac{c p_{\perp}}{e T} \sin \varphi$$

$$\rightarrow \boxed{x = \frac{c p_{\perp}}{e T} \sin \varphi, \quad y = \frac{c p_{\perp}}{e T} \cos \varphi} \quad \text{mit e mitlosky}$$

don origin

$$\text{But } \frac{d\varphi}{dt} = \frac{c\Omega}{r} = \frac{c\Omega}{\sqrt{T_0^2 + c^2\epsilon^2 t^2}}$$

$$\rightarrow \varphi = \frac{\Omega}{\epsilon} \operatorname{arsh} \left( \frac{c\epsilon t}{T_0} \right)$$

$$\text{check: } \frac{d\varphi}{dt} = \frac{\Omega}{\epsilon} \frac{c\epsilon}{T_0} \frac{1}{\sqrt{1 + \left(\frac{c\epsilon t}{T_0}\right)^2}} = \frac{c\Omega}{\sqrt{T_0^2 + c^2\epsilon^2 t^2}} \quad \checkmark$$

$$\rightarrow \operatorname{arsh} \frac{c\epsilon t}{T_0} = \operatorname{arsh} \left( \frac{c\epsilon t}{T_0} \right) = \sqrt{1 + c^2\epsilon^2 t^2 / T_0^2} = T_0 / T_0$$

$$\text{while b) } \rightarrow t = T_0 / c\epsilon$$

$$\rightarrow \underline{\underline{t = \frac{T_0}{c\epsilon} \operatorname{arsh} \frac{c\epsilon t}{T_0}}}$$

Now we have the orbit in a parametric representation:

$$\boxed{x = \frac{cp_{\perp}}{e\Omega} \sin \varphi, \quad y = \frac{cp_{\perp}}{e\Omega} \cos \varphi, \quad z = \frac{T_0}{c\epsilon} \operatorname{arsh} \frac{c\epsilon t}{T_0}}$$

When  $\varphi$  is related to  $t$  via

$$\boxed{\varphi = \frac{\Omega}{\epsilon} \operatorname{arsh} \frac{c\epsilon t}{T_0}} \quad \text{or} \quad \boxed{t = \frac{T_0}{c\epsilon} \operatorname{arsh} \frac{c\epsilon \varphi}{T_0}}$$

- d) The orbit is a helix whose pitch increases exponentially with increasing angle. Choosing the unit of length such that  $r_{\perp} := cp_{\perp}/e\Omega = 1$  we have

$$\boxed{x = \cos \varphi, \quad y = \sin \varphi, \quad z = z_0 \operatorname{arsh} (E\varphi/\Omega)} \quad \text{with } z_0 = \frac{T_0 \Omega}{cp_{\perp} \epsilon}$$

Here is an example for  $z_0 = 1$ ,  $E/\Omega = 0.1$ :

p-0.2.8-4

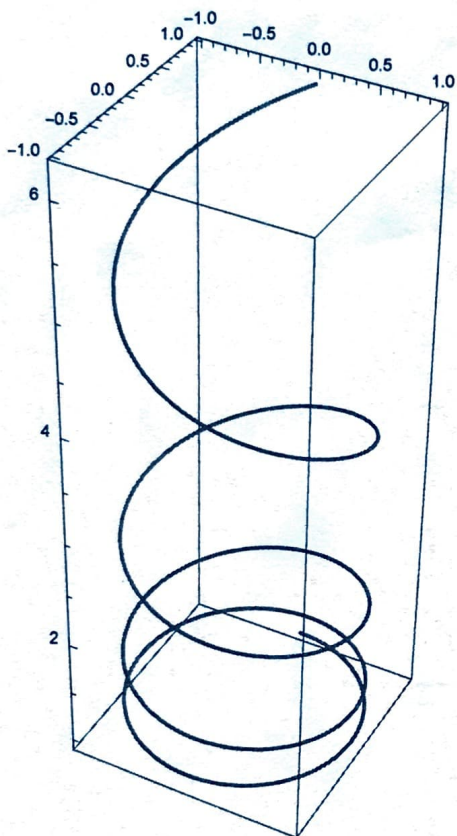
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In[198]= z0 = 1
          EB = 0.1
          x[phi_] := Sin[phi]
          y[phi_] := Cos[phi]
          z[phi_] := z0 Cosh[EB phi]
          ParametricPlot3D[{x[phi], y[phi], z[phi]}, {phi, 0, 8 Pi}]

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Out[198]= 1

Out[199]= 0.1



Out[203]=

①

For  $t \gg \frac{T_0}{cE}$  we have  $T \rightarrow cEt$

$$\rightarrow \dot{\varphi} = \frac{cE\beta}{T} \rightarrow \frac{cE\beta}{cEt} = \frac{\beta}{E} \frac{1}{t} \rightarrow 0 \quad \text{the angular velocity goes to zero}$$

①

$z(t) \rightarrow ct \rightarrow \dot{z} \rightarrow c$  the velocity in z-direction approaches c

For  $t \ll \frac{T_0}{cE}$  we have  $\dot{\varphi} = c\beta c / T_0 + O(t^1)$  w.st. angular velocity

$$\text{and } z(t) = z_0 + \frac{1}{2} \frac{c^2 e}{T_0} t^2 + O(t^4) \quad \text{Galilei rindt}$$

①

1.1.1.)  $\mathcal{L}$  is §1.1 def. 1  $\Rightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\begin{aligned} \Rightarrow \underline{\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu}} &= \partial^\lambda \partial^\mu A^\nu - \partial^\lambda \partial^\nu A^\mu \\ &+ \partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu \\ &+ \partial^\nu \partial^\lambda A^\mu - \partial^\nu \partial^\mu A^\lambda = 0 \quad (*) \end{aligned}$$

①

$$\Rightarrow \underline{\partial_\mu \tilde{F}^{\mu\nu}} = \epsilon^{\mu\nu\lambda\sigma} \partial_\mu F_{\lambda\sigma}$$

$$\text{relabelling} \Rightarrow \frac{1}{2} [\epsilon^{\mu\nu\lambda\sigma} \partial_\mu F_{\lambda\sigma} + \epsilon^{\lambda\nu\mu\sigma} \partial_\lambda F_{\mu\sigma} + \epsilon^{\lambda\mu\nu\sigma} \partial_\lambda F_{\sigma\mu}]$$

$$\epsilon \text{ symmetric} \Rightarrow \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} [\partial_\mu F_{\lambda\sigma} + \partial_\lambda F_{\sigma\mu} + \partial_\sigma F_{\mu\lambda}] \stackrel{(*)}{=} 0$$

①

1.1.2.)

$$S_{GL} = \int d\vec{x} \left[ r |\phi(\vec{x})|^2 + c |(\vec{\nabla} - iq\vec{A}(\vec{x}))\phi(\vec{x})|^2 + u |\phi(\vec{x})|^4 + \frac{1}{16\pi} F_{ij}(\vec{x}) F^{ij}(\vec{x}) \right]$$

c)  $0 \stackrel{!}{=} \frac{\delta S_{GL}}{\delta \phi(\vec{x})} = -r \phi(\vec{x}) + 2c \phi(\vec{x}) |\phi(\vec{x})|^2 + c \frac{\delta}{\delta \phi(\vec{x})} \int d\vec{x} (\vec{\nabla} - iq\vec{A})\phi (\vec{\nabla} - iq\vec{A})\phi$

$$= r\phi + 2c\phi|\phi|^2 - c \vec{\nabla}(\vec{\nabla} - iq\vec{A})\phi + iq\vec{A}(\vec{\nabla} - iq\vec{A})\phi$$

$$= -r\phi + 2c\phi|\phi|^2 - c(\vec{\nabla} - iq\vec{A})^2\phi$$

①

$0 \stackrel{!}{=} \frac{\delta S_{GL}}{\delta \vec{A}(\vec{x})} = c(-iq)\phi(\vec{\nabla} + iq\vec{A})\phi^* + c iq \left( (\vec{\nabla} - iq\vec{A})\phi \right) \phi^* + \frac{1}{16\pi} \frac{\delta}{\delta \vec{A}(\vec{x})} \int d\vec{y} F_{ij}(\vec{y}) F^{ij}(\vec{y})$

①

$$F_{ij} F^{ij} = (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) = 2 \varepsilon^{lijk} \varepsilon_{klmn} \partial_i A_j \partial^l A^m = 2(\vec{\nabla} \times \vec{A})^2$$

$$\rightarrow \frac{\delta}{\delta \vec{A}(\vec{x})} \int d\vec{y} F_{ij} F^{ij} = -\frac{\delta}{\delta \vec{A}(\vec{x})} \int d\vec{y} \vec{A}(\vec{y}) \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{y})))$$

$$= -2 \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x}))$$

①

$$= c iq \phi^* (\vec{\nabla} - iq\vec{A})\phi + c.c. - \frac{1}{4\pi} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x}))$$

$$\rightarrow -c [\vec{\nabla} - iq\vec{A}(\vec{x})]^2 \phi(\vec{x}) + [r + 2u |\phi(\vec{x})|^2] \phi(\vec{x}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x})) = 4\pi \mu c iq \phi^*(\vec{x}) [\vec{\nabla} - iq\vec{A}(\vec{x})] \phi(\vec{x}) + c.c.$$

①

GL eqs.



b) let  $\phi(\vec{x}) \rightarrow \phi(\vec{x}) e^{i g \lambda(\vec{x})}$ ,  $\vec{A}(\vec{x}) \rightarrow \vec{A}(\vec{x}) + \vec{\nabla} \lambda(\vec{x})$

$\rightarrow |\phi(\vec{x})|^2 \rightarrow |\phi(\vec{x})|^2$

and  $F_{ij}(\vec{x}) = \partial_i A_j - \partial_j A_i \rightarrow \partial_i (A_j + \partial_j \lambda) - \partial_j (A_i + \partial_i \lambda) = F_{ij}(\vec{x})$

①

Finally,

$$\begin{aligned} (\vec{\nabla} - i g \vec{A}) \phi &\rightarrow (\vec{\nabla} - i g \vec{A} - i g \vec{\nabla} \lambda) \phi e^{i g \lambda} = \\ &= (\vec{\nabla} \phi) e^{i g \lambda} + i g (\vec{\nabla} \lambda) \phi e^{i g \lambda} - i g \vec{A} \phi e^{i g \lambda} - i g (\vec{\nabla} \lambda) \phi e^{i g \lambda} \\ &= e^{i g \lambda} (\vec{\nabla} - i g \vec{A}) \phi \end{aligned}$$

$\rightarrow |(\vec{\nabla} - i g \vec{A}) \phi|^2 \rightarrow |(\vec{\nabla} - i g \vec{A}) \phi|^2$

①

 $\rightarrow S_{GL}$  is gauge invariantc) Modify  $\mathcal{L}$  from Problem 3b) to read

$$\mathcal{L} = (\partial_\mu \phi(x)) (\partial^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (*)$$

where  $\partial_\mu = \partial_\mu - i g A_\mu$

let  $\phi(x) \rightarrow \phi(x) e^{i g \lambda(x)}$ ,  $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \lambda(x)$

$\rightarrow |\phi|^2 \rightarrow |\phi|^2$  and  $F_{\mu\nu} F^{\mu\nu} \rightarrow F_{\mu\nu} F^{\mu\nu}$

$$\begin{aligned} \partial_\mu \phi &\rightarrow (\partial_\mu - i g A_\mu - i g \partial_\mu \lambda) \phi e^{i g \lambda} \\ &= e^{i g \lambda} (\partial_\mu + i g \partial_\mu \lambda - i g A_\mu - i g \partial_\mu \lambda) \phi = e^{i g \lambda} \partial_\mu \phi \end{aligned}$$

$\rightarrow (\partial_\mu \phi) (\partial^\mu \phi)^* \rightarrow (\partial_\mu \phi) (\partial^\mu \phi)^*$

①

 $\rightarrow (*)$  is gauge invariant