Problem Assignment # 4

01/29/2021
due 02/05/2021

1.2.1. Energy-momentum tensor

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

$$H_\mu{}^\nu(x) = (\partial_\mu A_\alpha(x))\,\frac{\partial\mathcal{L}}{\partial\,(\partial_\nu A_\alpha(x))} - \delta_\mu{}^\nu\,\mathcal{L}$$

obeys the continuity equation

$$\partial_\nu\,H_\mu{}^\nu(x) = 0 \quad (*)$$

note: Notice that $H_\mu{}^\nu(x)$ is a generalization of Jacobi's integral in Classical Mechanics.

b) Show that (*) also holds for

$$\tilde{T}_\mu{}^\nu = H_\mu{}^\nu + \partial_\alpha \psi_\mu{}^{\nu\alpha}$$

where $\psi_\mu{}^{\nu\alpha}$ is any tensor field that is antisymmetric in the second and third indices, $\psi_\mu{}^{\nu\alpha}(x) = -\psi_\mu{}^{\alpha\nu}(x)$.

c) Show that $\psi_\mu{}^{\nu\alpha}$ can be chosen such that $\tilde{T}_\mu{}^\nu(x) = T_\mu{}^\nu(x)$, which provides an alternative proof that $T_\mu{}^\nu(x)$ obeys (*).

(5 points)

1.2.2. Energy-momentum conservation in the presence of matter

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

$$\partial_\nu T_\mu{}^\nu(x) = \frac{-1}{c}\,F_\mu{}^\nu(x)\,J_\nu(x)$$

(2 points)

1.2.3. Energy-momentum tensor for a massive scalar field

Consider the massive scalar field from ch. 0 §2.5:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\varphi)\,(\partial^\mu\varphi) - \frac{m^2}{2}\,\varphi^2$$

and the tensor field $H_\mu{}^\nu$ defined analogously to Problem 1.2.1:

$$H_\mu{}^\nu = (\partial_\mu\varphi)\,\frac{\partial\mathcal{L}}{\partial\,(\partial_\nu\varphi)} - \delta_\mu{}^\nu\,\mathcal{L}$$

Determine $H_\mu{}^\nu$ explicitly and show that

$$\partial_\nu H_\mu{}^\nu = 0$$

hint: Use the Euler-Lagrange equation determined in ch. 0 §2.5.

(2 points)

... /over

1.2.4. **Coulomb gauge**

Consider the 4-vector potential $A^\mu(x) = (\varphi(x), \boldsymbol{A}(x))$. Show that one can always find a gauge transformation such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(x) = 0$$

This choice is called *Coulomb gauge*.

(2 points)

1.2.1.) a) $\partial_\nu \delta_\mu{}^\nu \mathcal{L} = \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial A_\alpha}\partial_\mu A_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu\partial_\lambda A_\alpha$

(1) $\overset{EL}{=} \left(\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\right)\partial_\mu A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu\partial_\lambda A_\alpha$

$= \partial_\lambda \frac{\partial\mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\partial_\mu A_\alpha$

(1) $\rightsquigarrow 0 = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)}\partial_\mu A_\alpha - \delta_\mu{}^\nu\mathcal{L}\right) = \partial_\nu \Theta_\mu{}^\nu$

b) $\partial_\nu\partial_\alpha\psi_\mu{}^{\nu\alpha} = -\partial_\nu\partial_\alpha\psi_\mu{}^{\alpha\nu} = -\partial_\alpha\partial_\nu\psi_\mu{}^{\nu\alpha} = -\partial_\nu\partial_\alpha\psi_\mu{}^{\nu\alpha}$

(1) $\rightsquigarrow \partial_\nu\partial_\alpha\psi_\mu{}^{\nu\alpha} = 0 \quad \rightsquigarrow \partial_\nu \tilde{T}_\mu{}^\nu = 0$

(1) c) Wähle $\psi^{\mu\nu\alpha} = \frac{1}{4\pi}A^\mu F^{\nu\alpha} = \frac{-1}{4\pi}A^\mu F^{\alpha\nu} = -\psi^{\mu\alpha\nu}$ ✓

$\rightsquigarrow \partial_\nu \tilde{T}_\mu{}^\nu = 0$, ok

$\tilde{T}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha\psi^{\mu\nu\alpha} = (\partial^\mu A_\alpha)\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)} - g^{\mu\nu}\mathcal{L} + \frac{1}{4\pi}\partial_\alpha A^\mu F^{\nu\alpha}$

$\overset{§1.3}{=} (\partial^\mu A_\alpha)\frac{1}{4\pi}F^{\alpha\nu} + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda} + \frac{1}{4\pi}(\partial_\alpha A^\mu)F^{\nu\alpha} + \frac{1}{4\pi}A^\mu\underbrace{\partial_\alpha F^{\nu\alpha}}_{=0}$

$= \frac{-1}{4\pi}(\partial^\mu A_\alpha - \partial_\alpha A^\mu)F^{\nu\alpha} + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda}$

$= \frac{-1}{4\pi}F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda}$

$= \frac{-1}{4\pi}F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{16\pi}g^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda} = T^{\mu\nu}$

(1) $\rightsquigarrow \partial_\nu T_\mu{}^\nu = 0$

1.22.) Generalizing the proof of the proposition in 1 §2.3:

The only difference is that now the EL eq. reads

1
$$\partial_\nu F^\nu{}_\alpha = \frac{4\pi}{c} j_\alpha$$

$$\leadsto \underline{\partial_\nu T_\mu{}^\nu} \overset{\S 2.3}{=} \frac{1}{4\pi}\left[-(\partial_\nu F_\mu{}^\alpha) F^\nu{}_\alpha - F_\mu{}^\alpha \partial_\nu F^\nu{}_\alpha + \frac{1}{4}\partial_\mu F_{\alpha\beta} F^{\alpha\beta}\right]$$

$$= -\frac{1}{c} F_\mu{}^\alpha j_\alpha + \frac{1}{4\pi}\underbrace{\left[-(\partial_\nu F_\mu{}^\alpha) F^\nu{}_\alpha + \frac{1}{2}(\partial_\mu F_{\alpha\beta}) F^{\alpha\beta}\right]}_{=0 \text{ by } \S 2.3}$$

1
$$= \underline{\underline{-\frac{1}{c} F_\mu{}^\nu j_\nu}}$$

1.2.3.) $\underline{H_\mu^{\ \nu}} = (\partial_\mu\varphi)\dfrac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)} - \delta_\mu^{\ \nu}\mathcal{L}$

$$= \underline{(\partial_\mu\varphi)(\partial^\nu\varphi) - \delta_\mu^{\ \nu}\tfrac{1}{2}(\partial_\alpha\varphi)(\partial^\alpha\varphi) + \delta_\mu^{\ \nu}\tfrac{m^2}{2}\varphi^2}$$

(1)

$\rightarrow \underline{\underline{\partial_\nu H_\mu^{\ \nu}}} = (\partial_\nu\partial_\mu\varphi)(\partial^\nu\varphi) + (\partial_\mu\varphi)(\partial_\nu\partial^\nu\varphi) - (\partial_\alpha\varphi)(\partial_\mu\partial^\alpha\varphi) + m^2\varphi\partial_\mu\varphi$

$$= \underbrace{(\partial_\nu\partial_\mu\varphi)(\partial^\nu\varphi) - (\partial_\mu\partial_\alpha\varphi)(\partial^\alpha\varphi)}_{=0} + (\partial_\mu\varphi)(\partial_\nu\partial^\nu\varphi + m^2\varphi)$$

$$= (\partial_\mu\varphi)\underbrace{(\partial_\nu\partial^\nu + m^2)\varphi}_{=0} = \underline{\underline{0}}$$

by the Klein-Gordon eq. (cf. p 2.5)

(1)

1.2.4.) Gauge trafo: $A^\mu \to A^\mu - \partial^\mu \chi$

$$\to \vec{A} \to \vec{A} - \vec{\nabla}\chi$$

$$\to \vec{\nabla}\cdot\vec{A} \to \vec{\nabla}\cdot\vec{A} - \vec{\nabla}^2\chi$$

(1)

Now choose χ as any solution of the Poisson eq.

$$\vec{\nabla}^2\chi(x) = \vec{\nabla}\cdot\vec{A}(x)$$

Then the transformed $\vec{A}$ has the property

$$\underline{\underline{\vec{\nabla}\cdot\vec{A}'(x)}} = \vec{\nabla}\cdot\vec{A}(x) - \vec{\nabla}^2\chi(x) = \underline{\underline{0}}$$

(1)