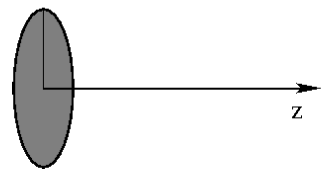
PHYS 622 | Classical Electrodynamics | W 2021

Problem Assignment # 7 | 02/19/2021
due 02/26/2021

2.2.1. **Planar charge distributions**

a) Consider a homogeneously charged infinitesimally thin ring with radius R and total charge Q that is oriented perpendicular to the z-axis. Calculate the electric field on the z-axis.



b) The same for a homogeneously charged disk with charge density σ and radius R. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

2.2.2. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\boldsymbol{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

   hint: Make an *ansatz* for a purely radial field, $\boldsymbol{E}(\boldsymbol{x}) = E(r)\,\hat{e}_r$, and integrate Gauss's law over a spherical volume.

Explicitly calculate and plot the field $\boldsymbol{E}(\boldsymbol{x})$ and the potential $\varphi(\boldsymbol{x})$ for

b) a homogeneously charged sphere

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if } \ r \leq r_0 \\ 0 & \text{if } \ r > r_0 \ . \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(\boldsymbol{x}) = \sigma_0\,\delta(r - r_0) \ .$$

(8 points)

2.2.3. **Electrostatics in d dimensions (to be continued later)**

Consider the third Maxwell equation in d dimensions:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{x}) = S_d\,\rho(\boldsymbol{x})$$

with the electric field $\boldsymbol{E}$ a d-vector, and S_d the area of the $(d-1)$-sphere: $S_{2n} = 2\pi^n/(n-1)!$ and $S_{2n+1} = 2^{2n+1} n!\pi^n/(2n)!$ for even and odd dimensions, respectively. Define a scalar potential $\varphi(\boldsymbol{x})$ in analogy to the $3-d$ case, such that

$$\boldsymbol{E}(\boldsymbol{x}) = -\boldsymbol{\nabla}\varphi(\boldsymbol{x})$$

and consider Poisson's equation

$$\boldsymbol{\nabla}^2\varphi(\boldsymbol{x}) = -S_d\,\rho(\boldsymbol{x})$$

note: Here we consider a generalization of electrostatics to d-dimensional space, NOT a d-dimensional charge distribution embedded in 3-dimensional space.

. . . **/over**

a) Show that the Green function $G_d(\boldsymbol{x})$ function for Poisson's equation, i.e., the solution of

$$\boldsymbol{\nabla}^2 G_d(\boldsymbol{x}) = -S_d\, \delta(\boldsymbol{x})$$

is given by

$$G_d(\boldsymbol{x}) = \frac{1}{d-2} \frac{1}{|\boldsymbol{x}|^{d-2}}$$

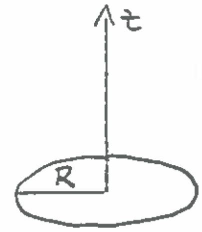
for all $d \neq 2$, and by

$$G_2(\boldsymbol{x}) = \ln(1/|\boldsymbol{x}|)$$

for $d = 2$.

hint: For $d = 1$, differentiate directly, using PHYS 610 Problem 36b). For $d \geq 2$, show that $G_d(\boldsymbol{x})$ is a harmonic function for all $\boldsymbol{x} \neq 0$, then integrate $\boldsymbol{\nabla}^2 G_d$ over a hypersphere around the origin and use Gauss's law.

(4 points)



16. a) Let the ring be in the $z=0$ plane:

$$\rho(\vec\xi) = \rho_0\, \delta(\xi_z)\, \delta(r-R)$$

in cylinder coordinates.

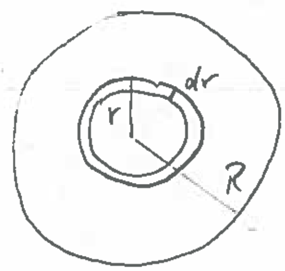
Total charge: $\int d\vec\xi\, \rho(\vec\xi) = 2\pi\rho_0 =: Q$

Poisson's formula: $\varphi(\vec x) = \int d\vec\xi\, \dfrac{\rho(\vec\xi)}{|\vec x - \vec\xi|}$

1 electric field: $\vec E = -\vec\nabla\varphi = -\int d\vec\xi\, \rho(\vec\xi)\, \vec\nabla \dfrac{1}{|\vec x-\vec\xi|} = \int d\vec\xi\, \rho(\vec\xi)\, \dfrac{\vec x-\vec\xi}{|\vec x-\vec\xi|^{3/2}}$

symmetry $\rightarrow$ $\vec E(\vec x = (0,0,z)) = E(z)\,\hat z$

1 $\rightarrow$ $E(z) = z \int d\vec\xi\, \dfrac{\rho(\vec\xi)}{|\vec x-\vec\xi|^{3/2}} = z \int_0^{2\pi} d\varphi\, \dfrac{\rho_0}{(z^2+R^2)^{3/2}} = \dfrac{Qz}{(z^2+R^2)^{3/2}}$



b) Charge density: σ

$\rightarrow$ charge of ring with radius r, thickness dr:

$$dQ = \sigma\, 2\pi r\, dr = \frac{Q}{\pi R^2}\, 2\pi r\, dr = \frac{2Q}{R^2}\, r\, dr \qquad Q = \pi R^2 \sigma$$

c) $\rightarrow$ $E(z) = \int_0^R dr\, \dfrac{2Q}{R^2}\, r\, \dfrac{z}{(z^2+r^2)^{3/2}} = \dfrac{2Q}{R^2}\, z\, \dfrac{1}{2} \int_0^{R^2} \dfrac{dx}{(z^2+x)^{3/2}}$

1 $= -\dfrac{2Q}{R^2}\, \dfrac{1}{(1+x)^{1/2}} \Big|_0^{R^2/z^2} = \dfrac{2Q}{R^2}\left(1 - \dfrac{z}{\sqrt{R^2+z^2}}\right) = 2\pi\sigma \left(1 - \dfrac{z}{\sqrt{R^2+z^2}}\right)$

$E(z\to\infty) = \dfrac{2Q}{R^2}\left(1 - 1 + \dfrac{1}{2}\dfrac{R^2}{z^2} + O(R^4/z^4)\right) = Q/z^2 + O(z^{-4})$ field of point charge.

$$E(z \to 0) = E(R \to \infty) = 2\pi\sigma$$

(1) An infinite sheet with constant charge density produces a field that's independent of z !

17.) a) Consider Gauss's law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

and integrate over a sphere with volume V:

$$\int_V d\vec{x}\, \vec{\nabla}\cdot\vec{E} = \int_{(V)} d\vec{s}\cdot\vec{E} = 4\pi \int_V d\vec{x}\, \rho$$

let $\rho(\vec{x})$ be spherically symmetric, $\rho(\vec{x}) = \rho(r)$, and make an

ansatz: $\boxed{\vec{E}(\vec{x}) = E(r)\hat{e}_r}$ $\quad \hat{e}_r = \vec{x}/|\vec{x}|$

$\rightsquigarrow \quad 4\pi r^2 E(r) = 4\pi \cdot 4\pi \int_0^r dr'\, r'^2 \rho(r')$

$\rightsquigarrow \quad \boxed{E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2 \rho(r')}$

(1)

For the potential, consider $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$

spherical symmetry $\rightsquigarrow \vec{\nabla}\varphi = \partial_r\varphi\, \hat{r}$

$\rightsquigarrow \quad E(r) = -\partial_r \varphi(r)$

$\rightsquigarrow \quad \varphi(r) = -\int_\infty^r dr'\, E(r')$ if we choose $\varphi(r=\infty) = 0$

$\rightsquigarrow \quad \boxed{\varphi(r) = \int_r^\infty dr'\, E(r')}$

(1)

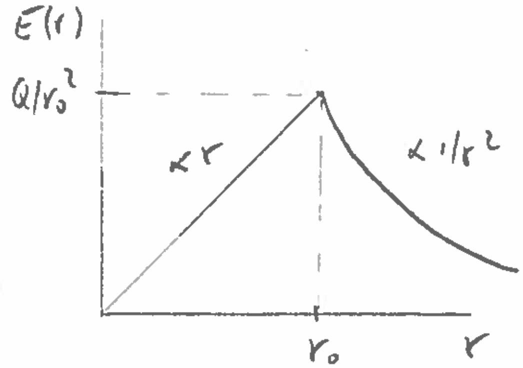
b) $E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2 \rho_0\, \Theta(r'<r_0)$

1st case: $r<r_0$ $\quad E(r) = \frac{4\pi\rho_0}{r^2}\int_0^r dx\, x^2 = \frac{4\pi\rho_0}{r^2}\frac{1}{3}r^3 = \frac{4\pi}{3}\rho_0 r$

$= \frac{4\pi}{3} r_0^3 \rho_0 \frac{r}{r_0^3} = \frac{Qr}{r_0^3}$ with $Q = \frac{4\pi}{3} r_0^3 \rho_0$ = total charge

2nd case: $r > r_0$ $\quad E(r) = \frac{4\pi}{r^2} \int_0^{r_0} dr' \, r'^2 \rho_0 = \frac{4\pi}{r^2} \rho_0 \frac{1}{3} r_0^3 = Q/r^2$

$$\rightarrow \quad \boxed{E(r) = \begin{cases} Qr/r_0^3 & \text{for } r \le r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}} \qquad \boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$$



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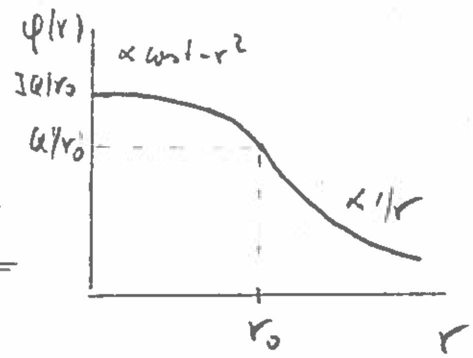
Now the potential:

1st case: $r < r_0$: $\quad \varphi(r) = \int_r^{r_0} dr' \frac{Qr'}{r_0^3} + \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r_0^3} \frac{1}{2}(r_0^2 - r^2) + \frac{Q}{r_0}$

$$= \frac{Q}{2r_0^3}(3r_0^2 - r^2)$$



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2nd case: $r > r_0$ $\quad \varphi(r) = \int_r^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r}$

$$\boxed{\varphi(r) = \begin{cases} \frac{Q}{2r_0^3}(3r_0^2 - r^2) & \text{for } r < r_0 \\ Q/r & \text{for } r \ge r_0 \end{cases}}$$

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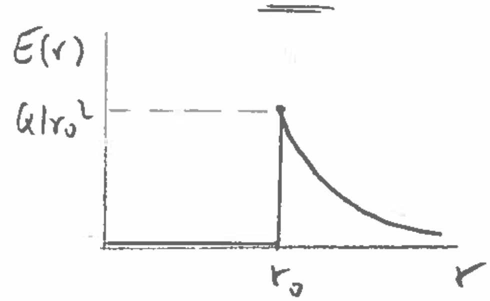
c) electric field: $\quad r < r_0$: $\quad E(r) = 0$

$r > r_0$ $\quad E(r) = \frac{4\pi}{r^2} \sigma_0 r_0^2 = \frac{Q}{r^2}$

$$\boxed{E(r) = \begin{cases} 0 & \text{for } r < r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$



with $Q = 4\pi r_0^2 \sigma_0$ = total charge

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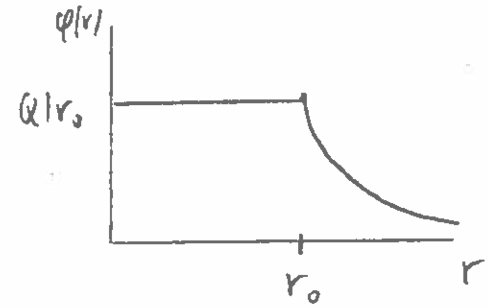
For $r > r_0$, $E(r)$ is the same as for the homogeneous sphere!

potential: $r < r_0$ $\quad \varphi(r) = \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = Q/r_0$

$r > r_0$ $\quad \varphi(r) = \int_{r}^{\infty} dr' \frac{Q}{r'^2} = Q/r$

$$\boxed{\varphi(r) = \begin{cases} Q/r_0 & \text{for } r < r_0 \\ Q/r & \text{for } r > r_0 \end{cases}}$$



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18. c) Consider $\vec\nabla^2 G_d(\vec x) = - S_d\, \delta(\vec x)$

with S_d the surface area of the $(d-1)$-sphere.

proposition: $G_d(\vec x) = \frac{1}{d-2}\,\frac{1}{|\vec x|^{d-2}}$ for $d \neq 2$

$G_d(\vec x) = \log\left(1/|\vec x|\right)$ for $d = 2$

proof: $d=1$ by direct differentiation: 610 Problem 36 b)

$$\frac{d^2}{dx^2}(-)|x| = -\frac{d}{dx}\,\mathrm{sgn}\, x \overset{\downarrow}{=} -2\,\delta(x)$$

$$\rightsquigarrow \frac{d^2}{dx^2} G_{d=1}(x) = \frac{d^2}{dx^2}(-)|x| = -2\,\delta(x) = - S_{d=1}\,\delta(x) \quad ✓$$

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$r = |\vec x|$

$d=2$: $\partial_i\partial_j \log|\vec x| = \partial_i \frac{x_j}{r^2} = \frac{r^2\delta_{ij} - x_j \frac{2x_i}{r} r}{r^4} = \frac{r^2\delta_{ij} - 2x_i x_j}{r^4}$

$\rightsquigarrow \vec\nabla^2 \log|\vec x| = \partial_i\partial^i \log|\vec x| = (2-2)\frac{1}{r^2} = 0 \quad \forall\, r \neq 0$

$\rightsquigarrow \log|\vec x|$ is a harmonic fct. $\forall\, \vec x \neq 0$

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Now integrate over a circle C_0 radius r_0:

$$\int_{C_0} d^2x\, \vec\nabla^2 \log|\vec x| = \int_{C_0} d^2x\, \vec\nabla\cdot(\vec\nabla \log r) \overset{\text{Gauss}}{=} \int_{(C_0)} d\vec\sigma\cdot\vec\nabla \log r$$

$$= \int_0^{2\pi} d\varphi\, r_0\, \frac{\vec x}{r}\cdot\frac{\vec x}{r^2}\Big|_{r_0} = 2\pi$$

$$\rightsquigarrow \vec\nabla^2 \log|\vec x| = 2\pi\,\delta(\vec x) = S_{d=2}\,\delta(\vec x)$$

$$\rightsquigarrow G_{d=2}(\vec x) = -\log|\vec x| = \log\left(1/|\vec x|\right)$$

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$\underline{d>2}$: $$\partial_i\partial_j \frac{1}{|\vec{x}|^{d-2}} = -\frac{d-2}{2}\partial_i \frac{2x_j}{|\vec{x}|^d} = -(d-2)\left(\frac{\delta_{ij}}{|\vec{x}|^d} - \frac{d}{2}\frac{2x_i x_j}{|\vec{x}|^{d+2}}\right)$$

$$= -(d-2)\,\frac{r^2\delta_{ij} - d x_i x_j}{r^{(d+2)}}$$

$$\rightsquigarrow \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}} = -(d-2)(d-d)\frac{1}{r^d} = 0 \quad \forall\, r \neq 0$$

Integrate over a hypersphere S_0 with radius r_0:

$$\underline{\int_{S_0} d^d x\, \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}}} = \int_{(S_0)} d\vec{\sigma}\cdot\vec{\nabla}\frac{1}{|\vec{x}|^{d-2}} = S_d\, r_0^{d-1}(-)(d-2)\frac{\vec{x}}{r}\cdot\frac{\vec{x}}{r^{d-1}}$$

$$= \underline{-(d-2)\,S_d}$$

$$\rightsquigarrow \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}} = -(d-2)\,S_d\,\delta(\vec{x})$$

(1) $$\rightsquigarrow \underline{\underline{G_{d>2}(\vec{x})}} = \underline{\underline{\frac{1}{d-2}\frac{1}{|\vec{x}|^{d-2}}}}$$

b) It is easiest to start with the field. Gauss's law in 2-d reads

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = 2\pi\,\rho(\vec{x})$$

and proceeding as in Problem 2.1.2 we have

$$2\pi r\, E(r) = 2\pi\cdot 2\pi\int_0^r dr'\, r'\,\rho(r')$$

for a charge distribution $\rho(\vec{x}) = \rho(r)$ and $\vec{E}(\vec{x}) = E(r)\,\hat{e}_r$.

(1) $$\rightsquigarrow \boxed{E(r) = \frac{2\pi}{r}\int_0^r dr'\, r'\,\rho(r')}$$

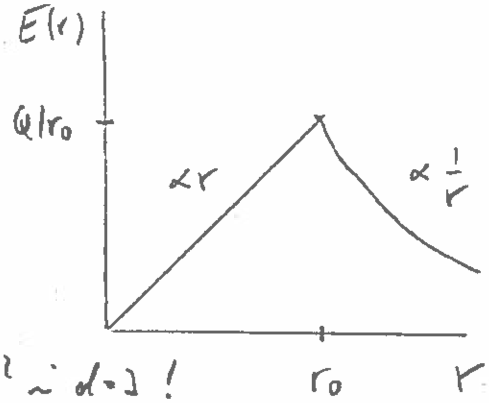
Homogeneously charged disk: $\rho(r) = \rho_0\,\Theta(r_0 - r)$

1st case: $r<r_0$ $\quad E(r) = \frac{2\sigma}{r}\int_0^r dr' r' \rho_0 = \frac{2\sigma\rho_0}{r}\frac{1}{2}r^2 = \sigma\rho_0 r$

$= \frac{Q}{r_0^2} r$ with $Q = \sigma r_0^2 \rho_0$ total charge

2nd case: $r>r_0$ $\quad E(r) = \frac{2\sigma}{r}\rho_0 \frac{1}{2} r_0^2 = \frac{Q}{r}$

$$\rightarrow \boxed{\vec E(\vec x) = E(r)\,\hat e_r \qquad E(r) = \begin{cases} Qr/r_0^2 & \text{for } r<r_0 \\ Q/r & \text{for } r>r_0 \end{cases}}$$



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Field falls off only as $1/r$, as opposed to $1/r^2$ in $d=3$!

Now the potential: $\quad \vec E(\vec x) = -\vec\nabla\varphi(\vec x) = -\partial_r \varphi(r)\,\hat e_r$

$\rightarrow E(r) = -\partial_r\varphi(r)$

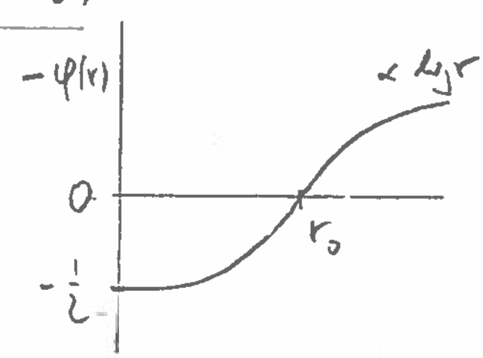
$\rightarrow \varphi(r) = -\int_{\text{const}}^{r} dr' E(r')$

$\rightarrow \boxed{\varphi(r) = -\int_{r_0}^{r} dr' E(r')}$ with the choice $\varphi(r=r_0)=0$

1st case: $r<r_0$ $\quad -\varphi(r) = -\int_r^{r_0} dr' \frac{Qr'}{r_0^2} = -\frac{Q}{2r_0^2}(r_0^2-r^2) = \frac{+Q}{2}\left(\frac{r^2}{r_0^2}-1\right)$

2nd case: $r>r_0$ $\quad -\varphi(r) = \int_{r_0}^{r} dr' \frac{Q}{r'} = Q\log(r/r_0)$

$$\boxed{-\varphi(r) = Q \times \begin{cases} \frac{1}{2}(r^2/r_0^2-1) & \text{for } r<r_0 \\ \log(r/r_0) & \text{for } r>r_0 \end{cases}}$$



(1)

(1) for sketches

NB: This is minus φ!

c) In 1-d it is easiest to integrate Poisson's formula directly:

$$\varphi(x) = \int dy\, G_{d=1}(x-y)\,\rho(y) = -\int dy\, |x-y|\, \rho_0\, \Theta(x_0^2/4 - y^2)$$

$$= -\rho_0 \int_{-x_0/2}^{x_0/2} dy\, |x-y| = \varphi(-x)$$

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Let $x \geq 0$.

1st case: $x < x_0/2$ $\quad \varphi(x) = -\rho_0 \int_{-x_0/2}^{x} dy\,(x-y) + \rho_0 \int_{x}^{x_0/2} dy\,(x-y)$

$$= -\rho_0\left[x\left(\cancel{x} + \tfrac{x_0}{2}\right) - \tfrac{1}{2}\left(\cancel{x^2} - \tfrac{1}{4}x_0^2\right)\right] + \rho_0\left[x\left(\tfrac{x_0}{2} - \cancel{x}\right) - \tfrac{1}{2}\left(\tfrac{x_0^2}{4} - \cancel{x^2}\right)\right]$$

$$= -\rho_0\left(x^2 + \tfrac{1}{4}x_0^2\right)$$

2nd case: $x > x_0/2$ $\quad \varphi(x) = -\rho_0 \int_{-x_0/2}^{x_0/2} dy\,(x-y) = -\rho_0\, x x_0 = -\rho_0 x_0 x$

$$= -Qx \quad \text{with } Q = \rho_0 x_0 \text{ total charge}$$

$$\boxed{\varphi(x) = -Q x_0 \times \begin{cases} \dfrac{x^2}{x_0^2} + \dfrac{1}{4} & \text{for } |x| < x_0/2 \\ |x|/x_0 & \text{for } |x| > x_0/2 \end{cases}}$$

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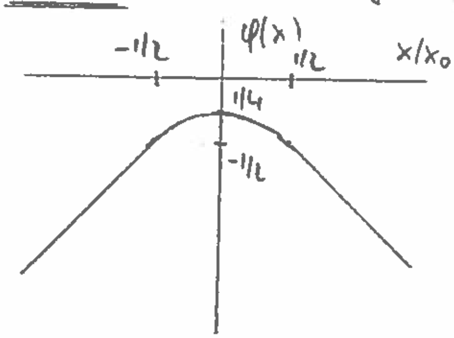

Now the field:

$$E(x) = -\partial_x \varphi(x) =$$

$$\boxed{E(x) = Q \times \begin{cases} 2x/x_0 & \text{for } |x| < x_0/2 \\ \operatorname{sgn} x & \text{for } |x| > x_0/2 \end{cases}}$$

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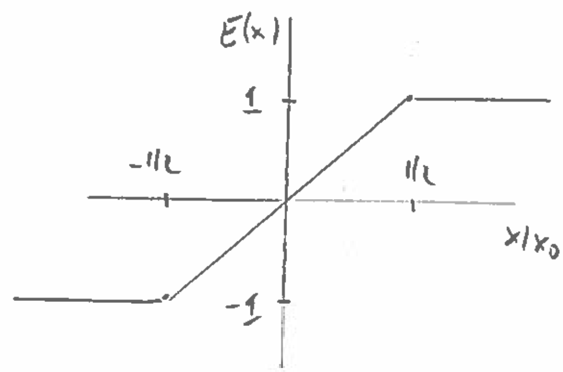

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Field does not fall off for $|x| \to \infty$!