Problem Assignment # 8

02/26/2021
due 03/05/2021

2.2.3. **Electrostatics in d dimensions (continued)**

This is a continuation of Problem #2.2.3.

b) Calculate and plot the potential φ and the field $\boldsymbol{E}$ for $d = 2$ for the case of a homogeneously charged disk, $\rho(\boldsymbol{x}) = \rho_0\,\Theta(r_0 - |\boldsymbol{x}|)$.

   hint: It is easiest to proceed as in the 3-d case, see Problem 2.2.2.

   note: This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0\,\Theta(x_0^2/4 - x^2)$.

   hint: Integrate Poisson's formula directly. (8 points)

2.2.4. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \boldsymbol{\nabla}^2)\varphi(\boldsymbol{x}) = 4\pi\rho(\boldsymbol{x})$$

in terms of an integral.
hint: The answer is a generalization of Poisson's formula.

(3 points)

2.3.1. **Quadrupole moments (to be continued later)**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r}\,Q + \frac{1}{r^3}\,\boldsymbol{x}\cdot\boldsymbol{d} + \frac{1}{r^5}\sum_{i,j} x_i x_j Q_{ij} + \ldots$$

   with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

(7 points)

2.2.3.) b) It is easiest to start with the field. Gauss's law in 2-d reads

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = 2\pi\,\rho(\vec{x})$$

and proceeding as in Problem 2.1.2 we have

$$2\pi r\,E(r) = 2\pi\cdot 2\pi\int_0^r dr'\,r'\,\rho(r')$$

for a charge distribution $\rho(\vec{x}) = \rho(r)$ and $\vec{E}(\vec{x}) = E(r)\,\hat{e}_r$.

$$\rightsquigarrow \boxed{E(r) = \frac{2\pi}{r}\int_0^r dr'\,r'\,\rho(r')}$$

(1)

Homogeneously charged disk: $\rho(r) = \rho_0\,\Theta(r_0 - r)$

1st case: $r<r_0$ $\quad E(r) = \frac{2\pi}{r}\int_0^r dr'\, r' \rho_0 = \frac{2\pi\rho_0}{r}\frac{1}{2}r^2 = \pi\rho_0 r$

$= \frac{Q}{r_0^2} r$ with $Q = \pi r_0^2 \rho_0$ total charge

2nd case: $r>r_0$ $\quad E(r) = \frac{2\pi}{r}\rho_0 \frac{1}{2} r_0^2 = \frac{Q}{r}$

$$\rightarrow \boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r \qquad E(r) = \begin{cases} Qr/r_0^2 & \text{for } r<r_0 \\ Q/r & \text{for } r>r_0 \end{cases}}$$

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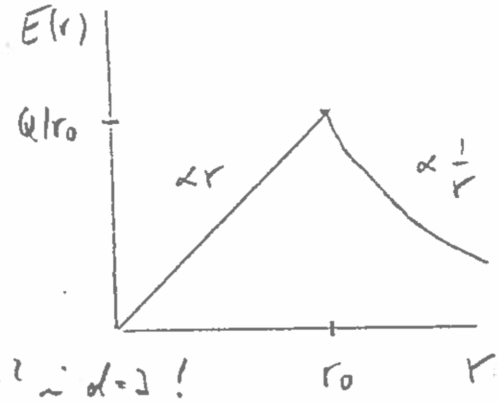

Field falls off only as $1/r$, as opposed to $1/r^2$ in $d=3$!

Now the potential: $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x}) = -\partial_r \varphi(r)\,\hat{e}_r$

$\rightarrow E(r) = -\partial_r \varphi(r)$

$\rightarrow \varphi(r) = -\int_{\text{const}}^{r} dr'\, E(r')$

$\rightarrow \boxed{\varphi(r) = -\int_{r_0}^{r} dr'\, E(r')}$ with the choice $\varphi(r=r_0)=0$

1st case: $r<r_0$ $\quad -\varphi(r) = -\int_r^{r_0} dr'\, \frac{Qr'}{r_0^2} = -\frac{Q}{2r_0^2}(r_0^2 - r^2) = \frac{+Q}{2}\left(\frac{r^2}{r_0^2}-1\right)$

2nd case: $r>r_0$ $\quad -\varphi(r) = \int_{r_0}^{r} dr'\, \frac{Q}{r'} = Q\log(r/r_0)$

$$\boxed{-\varphi(r) = Q \times \begin{cases} \frac{1}{2}(r^2/r_0^2 - 1) & \text{for } r<r_0 \\ \log(r/r_0) & \text{for } r>r_0 \end{cases}}$$

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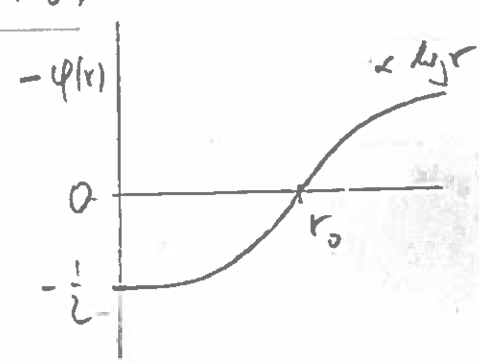


$\rightarrow$ [illegible]

c) In 1-d it is easiest to integrate Poisson's formula directly:

$$\underline{\underline{\varphi(x)}} = \int dy\, G_{d=1}(x-y)\,\rho(y) = -\int dy\, |x-y|\, \rho_0\, \Theta(x_0^2/4 - y^2)$$

$$= \underline{-\rho_0 \int_{-x_0/2}^{x_0/2} dy\, |x-y|} = \underline{\varphi(-x)}$$

(1)

Let $x \geq 0$.

$\underset{\sim}{1^{st}\text{ case}}$: $\underline{x < x_0/2}$ $\quad \underline{\varphi(x)} = -\rho_0 \int_{-x_0/2}^{x} dy\,(x-y) + \rho_0 \int_{x}^{x_0/2} dy\,(x-y)$

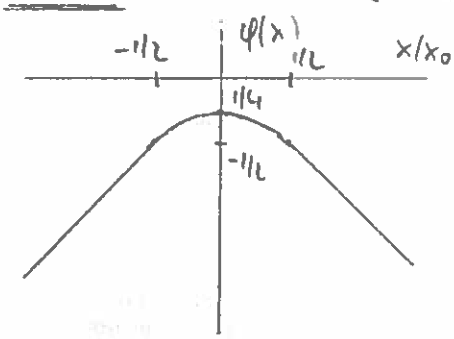
$$= -\rho_0 \left[x\left(\cancel{x} + \tfrac{x_0}{2}\right) - \tfrac{1}{2}\left(\cancel{x^2} - \tfrac{1}{4}x_0^2\right)\right] + \rho_0\left[x\left(\tfrac{x_0}{2} - \cancel{x}\right) - \tfrac{1}{2}\left(\tfrac{x_0^2}{4} - x^2\right)\right]$$

$$= \underline{-\rho_0\left(x^2 + \tfrac{1}{4}x_0^2\right)} = \ldots$$

$\underset{\sim}{2^{nd}\text{ case}}$: $\underline{x > x_0/2}$ $\quad \underline{\varphi(x)} = -\rho_0 \int_{-x_0/2}^{x_0/2} dy\,(x-y) = -\rho_0\, x x_0 = -\rho_0 x_0\, x$

$$= \underline{-Qx} \quad \text{with } \underline{Q = \rho_0 x_0} \text{ total charge}$$

$$\boxed{\varphi(x) = -Qx_0 \times \begin{cases} \dfrac{x^2}{x_0^2} + \dfrac{1}{4} & \text{for } |x| < x_0/2 \\[2mm] |x|/x_0 & \text{for } |x| > x_0/2 \end{cases}}$$

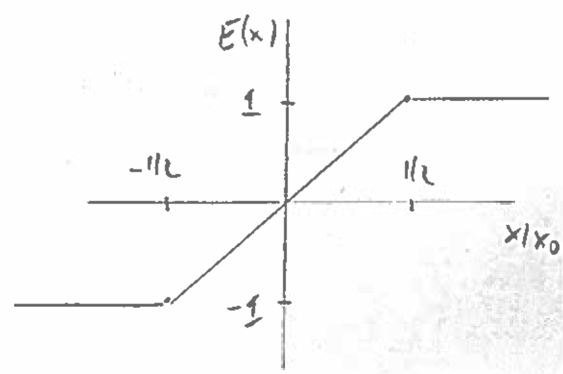


(1)

Now the field:

$$E(x) = -\partial_x \varphi(x) =$$

$$\boxed{E(x) = Q \times \begin{cases} 2x/x_0 & \text{for } |x| < x_0/2 \\ \operatorname{sgn} x & \text{for } |x| > x_0/2 \end{cases}}$$



(1)

(1)

Field does not fall off for $|x| \to \infty$!

2.2.4:) Helmholtz eq: $(\vec{\kappa}^2 - \vec{\nabla}^2)\varphi(\vec{x}) = 4\pi\rho(\vec{x})$

Fourier transf as in ch 3 § 2:

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$$(\vec{\kappa}^2 + \vec{k}^2)\,\hat{\varphi}(\vec{k}) = 4\pi\,\hat{\rho}(\vec{k})$$

$$\rightarrow \hat{\varphi}(\vec{k}) = \frac{4\pi}{\kappa^2 + \vec{k}^2}\,\hat{\rho}(\vec{k})$$

$$\rightarrow \varphi(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\vec{x}}\, \frac{4\pi}{\kappa^2 + \vec{k}^2}\,\hat{\rho}(\vec{k})$$

$$= \int d\vec{y}\; v_{sc}(\vec{x} - \vec{y})\,\rho(\vec{y})$$ by the convolution theorem, PHYS 610 ch 2 § 7.1

1

where $v_{sc}(\vec{x})$ is the Fourier back transf of the screened Coulomb potential

$$\hat{v}_{sc}(\vec{k}) = \frac{4\pi}{\kappa^2 + \vec{k}^2}$$

610 Problem 27 b) $\rightarrow$ $v_{sc}(\vec{x}) = \frac{1}{r}\,e^{-\kappa r}$ with $r = |\vec{x}|$

$$\rightarrow \boxed{\varphi(\vec{x}) = \int d\vec{y}\; \frac{e^{-\kappa|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\,\rho(\vec{y})}$$

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For $\kappa = 0$ we recover Poisson's formula

2.3.1.) a) $\frac{1}{|\vec x-\vec y|} = \frac{1}{r}\left(1-2\frac{\vec x\cdot\vec y}{r^2}+\frac{y^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1+\frac{\vec x\cdot\vec y}{r^2}-\frac{1}{2}\frac{y^2}{r^2}+\frac{3}{2}\frac{(\vec x\cdot\vec y)^2}{r^4}+\dots\right)$

$$= \frac{1}{r}\left[1+\frac{\vec x\cdot\vec y}{r^2}+\frac{3}{2}x_i x_j y_i y_j\frac{1}{r^4}-\frac{1}{2}y^2\delta_{ij}x_i x_j\frac{1}{r^4}+\dots\right]$$

$$= \frac{1}{r}\left[1+\frac{\vec x\cdot\vec y}{r^2}+\frac{1}{2}x_i x_j\left(3y_i y_j-\delta_{ij}y^2\right)\frac{1}{r^4}+\dots\right]$$

$$\rightsquigarrow \underline{\underline{\varphi(\vec x)}} = \int d\vec y\,\frac{\rho(\vec y)}{|\vec x-\vec y|} = \frac{1}{r}\int d\vec y\,\rho(\vec y)+\frac{1}{r^3}\vec x\cdot\int d\vec y\,\vec y\,\rho(\vec y)$$

$$+\frac{1}{2}\frac{1}{r^5}x_i x_j\int d\vec y\left(3y_i y_j-\delta_{ij}y^2\right)\rho(\vec y)+\dots$$

$$= \underline{\underline{\frac{1}{r}Q+\frac{1}{r^3}\vec x\cdot\vec d+\frac{1}{r^5}\sum_{ij}x_i x_j Q_{ij}+O(1/r^4)}}$$

where $Q = \int d\vec y\,\rho(\vec y)$ monopole moment

$\vec d = \int d\vec y\,\vec y\,\rho(\vec y)$ dipole moment

(1) $\underline{\underline{Q_{ij} = \frac{1}{2}\int d\vec y\left(3y_i y_j-\delta_{ij}y^2\right)\rho(\vec y)}}$ quadrupole moment

b) $\rho'(\vec y) = \rho(\vec y-\vec a)$

$$\rightsquigarrow \underline{\underline{Q'_{ij}}} = \frac{1}{2}\int d\vec y\left(3y_i y_j-\delta_{ij}y^2\right)\rho'(\vec y)$$

$$= \frac{1}{2}\int d\vec y\left[3(y_i+a_i)(y_j+a_j)-\delta_{ij}(\vec y+\vec a)^2\right]\rho(\vec y)$$

$$= Q_{ij}+\frac{1}{2}\int d\vec y\left[3a_i y_j+3a_j y_i+3a_i a_j-\delta_{ij}\left(2\vec a\cdot\vec y+\vec a^2\right)\right]\rho(\vec y)$$

$$= Q_{ij}+\frac{3}{2}a_i d_j+\frac{3}{2}a_j d_i+\frac{3}{2}a_i a_j Q-\delta_{ij}\,\vec a\cdot\vec d-\delta_{ij}\frac{1}{2}\vec a^2 Q$$

(1) $= \underline{\underline{Q_{ij}}}$ if $\underline{\vec d = Q = 0}$

c) ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$

$$\to \quad Q_{ij} = \frac{1}{2} \int d\vec{x} \, (3x_i x_j - \delta_{ij} \vec{x}^2) \, \Theta(x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1) \, \rho$$

where $\vec{x} = (x, y, z)$ and ρ = charge density

(1) symmetry $\to$ $Q_{ij} = 0$ unless $i = j$

$$Q_{11} = \frac{\rho}{2} \int dx\,dy\,dz \, (2x^2 - y^2 - z^2) \, \Theta(x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1)$$

$$= \frac{1}{2} \rho\, abc \int dx\,dy\,dz \, (2a^2x^2 - b^2y^2 - c^2z^2) \, \Theta(x^2 + y^2 + z^2 \le 1)$$

$$= \frac{1}{2} \rho\, abc \Big[2a^2 \int_0^1 dr\, r^2 \int_{-1}^{1} d\eta \int_0^{2\pi} d\varphi \, r^2 \sin^2\vartheta \cos^2\varphi$$
$$- b^2 \int_0^1 dr\, r^2 \int_{-1}^{1} d\eta \int_0^{2\pi} d\varphi \, r^2 \sin^2\vartheta \sin^2\varphi$$
$$- c^2 \int_0^1 dr\, r^2 \int_{-1}^{1} d\eta \int_0^{2\pi} d\varphi \, r^2 \cos^2\vartheta \Big]$$

$$= \frac{1}{2} \rho\, abc \left[2a^2 \cancel{\pi} \left(2 - \frac{2}{3}\right) \frac{1}{\cancel{5}} - b^2 \cancel{\pi} \left(2 - \frac{2}{3}\right) \frac{1}{\cancel{5}} - c^2 2\cancel{\pi} \frac{2}{3} \frac{1}{\cancel{5}} \right]$$

$$= \frac{1}{2} \rho\, abc \frac{\pi}{5} \left[\frac{8}{3} a^2 - \frac{4}{3} b^2 - \frac{4}{3} c^2 \right]$$

$$= \frac{4\pi}{3} abc\, \rho \, \frac{1}{10} \left[2a^2 - b^2 - c^2 \right]$$

(1)
$$= Q \, \frac{1}{10} (2a^2 - b^2 - c^2)$$ with $Q = \frac{4\pi}{3} \rho\, abc$ = total charge

$$Q_{22} = Q \, \frac{1}{10} (2b^2 - a^2 - c^2)$$ by symmetry

$$Q_{33} = Q \, \frac{1}{10} (2c^2 - b^2 - a^2)$$ by symmetry

(1) check: $Q_{11} + Q_{22} + Q_{33} = 0$ ✓

d) As a real symmetric tensor, Q_{ij} can always be diagonalized

$\rightarrow$ The most general form of Q_{ij} in its principal axes system is

$$Q_{ij} = \begin{pmatrix} q_+ + q_- & 0 & 0 \\ 0 & q_+ - q_- & 0 \\ 0 & 0 & -2q_+ \end{pmatrix}$$

where

$$\underline{\underline{q_-}} = \frac{1}{2}\left(Q_{11} - Q_{22}\right) = \frac{1}{2}\int d\vec{x}\, \rho(\vec{x})\, \left[2x^2 - y^2 - z^2 - 2y^2 + x^2 + z^2\right]$$

$$= \underline{\frac{3}{2}\int d\vec{x}\, \rho(\vec{x})\, (x^2 - y^2)}$$

(1)

cylindrical coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$ $\quad x^2 - y^2 = r^2(\cos^2\varphi - \sin^2\varphi) = r^2\cos 2\varphi$

$$\rightarrow q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r \int dz\, \rho(r,\varphi,z)\, r^2 \cos 2\varphi$$

Now let $\rho(r,\varphi,z) = \rho(r,\varphi+\alpha,z)$

$$\rightarrow q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi+\alpha,z)\cos 2\varphi$$

$$= \frac{3}{2}\int_\alpha^{2\pi+\alpha} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z)\cos 2(\varphi-\alpha)$$

$$= \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z)\left(\cos 2\varphi\cos 2\alpha + \sin 2\varphi \sin 2\alpha\right)$$

$r^2\sin 2\varphi = r^2\sin\varphi\cos\varphi = xy$ $\rightarrow$ the second term is $\propto Q_{12} = 0$

$$= \cos 2\alpha \cdot q_- \quad \rightarrow \quad \underline{\underline{q_- = 0}} \text{ since } \alpha \neq \pi$$

(1)

part c) with $a = b$ $\rightarrow$ $Q_{11} = Q_{22} = \frac{Q}{10}(a^2 - c^2)$ ✓

e)

$$\underline{\underline{Q_{20}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{2}(3\xi^2-1)$$

$$= \frac{1}{2}\int d\vec{x}\, \rho(\vec{x})\,(3z^2-r^2) = \underline{\underline{D_{33}}}$$

(1)

$$\underline{\underline{Q_{2,\pm 1}}} \propto \int d\Omega\, \underbrace{\underbrace{\rho(r,\Omega)}_{\text{even}}\; \underbrace{P_2^{\pm 1}(\xi)}_{\text{odd}}}_{\text{fct. of } \xi} = \underline{\underline{0}} \quad \text{by symmetry}$$

(1)

$$\underline{\underline{Q_{22}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{\sqrt{4!}}\, e^{2i\varphi}\, 3(1-\xi^2)$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (1-\xi^2)(\cos 2\varphi + i \sin 2\varphi)$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 \sin^2\theta\,(\cos^2\varphi - \sin^2\varphi) + \frac{3i}{\sqrt{24}} \int d\vec{x}\, \underbrace{\underbrace{\rho(\vec{x})\, r^2 \sin^2\theta}_{\text{even}}\, \underbrace{\sin 2\varphi}_{\text{odd}}}_{\text{fct. of } \varphi \;\to 0}$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (\sin^2\theta \cos^2\varphi - \sin^2\theta \sin^2\varphi)$$

$$\left.\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} \rightsquigarrow x^2 - y^2 = r^2 \sin^2\theta\,(\cos^2\varphi - \sin^2\varphi)$$

$$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x})\,(x^2-y^2)$$

$$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x}) \left[(2x^2-y^2-z^2) - (2y^2-x^2-z^2)\right]\frac{1}{3}$$

$$= \underline{\underline{\frac{1}{\sqrt{6}}\,(D_{11} - D_{22})}}$$

$$\underline{\underline{Q_{2,-2}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega)\, \cancel{\sqrt{\frac{5}{4\pi}}}\, \sqrt{4!}\, e^{-2i\varphi}\, \frac{1}{8}(1-\xi^2) = \underline{\underline{Q_{22}}}$$

(1)