PHYS 622 Classical Electrodynamics W 2021

Problem Assignment # 9 03/05/2021
due 03/12/2021

2.3.2. **Legendre polynomials**

Consider the ODE

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

with λ a constant. Show that a necessary condition for the existence of a polynomial solution is

$$\lambda = n(n+1)$$

with $n = 0, 1, \ldots$ What else do you need to require in order to get a condition that is necessary and sufficient? Convince yourself that these considerations correctly produce the first three Legendra polynomials up to an overall normalization factor.

hint: Make a power-series ansatz and require that the series terminates.

(4 points)

2.3.3. **Associated Legendre functions**

note: When comparing with the reference book by Abramowitz and Stegun, note that their $P_\ell^m(x)$ equals $(-)^{3m/2}$ times our $P_\ell^m(x)$.

Show that

$$\left(\sqrt{1-x^2}\,\frac{d}{dx} - m\,\frac{x}{\sqrt{1-x^2}}\right) P_\ell^m(x) = (\ell+m)(\ell-m+1)\,P_\ell^{m-1}(x)$$

hint: First differentiate Legendre's ODE $m-1$ times to show that

$$(1-x^2)\,\frac{d^{m+1}}{dx^{m+1}}\,P_n(x) - 2mx\,\frac{d^m}{dx^m}\,P_n(x) + (n+m)(n-m+1)\,\frac{d^{m-1}}{dx^{m-1}}\,P_n(x) = 0$$

Then use this in evaluating $\sqrt{1-x^2}\,dP_\ell^m(x)/dx$.

(3 points)

. . . **/over**

2.3.4. **Spherical harmonics**

Prove that the sperical harmonics have the following properties:

$$
\begin{aligned}
Y_\ell^m(\Omega)^* &= (-)^m Y_\ell^{-m}(\Omega) && (1)\\
\cos\theta\, Y_\ell^m(\Omega) &= \left(\frac{(\ell+1-m)(\ell+1+m)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^m(\Omega) + \left(\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^m(\Omega) && (2)\\
\sin\theta\, e^{\pm i\varphi}\, Y_\ell^m(\Omega) &= \pm\left(\frac{(\ell\mp m-1)(\ell\mp m)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^{m\pm 1}(\Omega) \mp \left(\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^{m\pm 1}(\Omega) && (3)\\
\hat{L}_\mp\, Y_\ell^m(\Omega) &= ((\ell\pm m)(\ell\mp m+1))^{1/2}\, Y_\ell^{m\mp 1}(\Omega) && (4)
\end{aligned}
$$

where

$$\hat{L}_\mp = e^{\mp i\varphi}\left[\mp\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right]$$

hint: Use the properties of the associated Legendre functions we quoted in ch.3 §3.2, as well as Problem 2.3.3.

(9 points)

2.3.5. **Field due to distant charges**

Consider the electric field generated by a charge density $\rho(\boldsymbol{y})$ that vanishes inside a sphere with radius r_0: $\rho(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \leq r_0$. Show that

a) If ρ is invariant under parity operations, $\rho(-\boldsymbol{y}) = \rho(\boldsymbol{y})$, then the electric field at the origin vanishes.

b) If $\rho(\boldsymbol{y})$ is invariant under rotations about the z-axis through multiples of an angle α with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form $\varphi_{ij}(\boldsymbol{x}=0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$

c) If $\rho(\boldsymbol{y})$ has cubic symmetry, i.e., if $\rho(\boldsymbol{y})$ is invariant under rotations through $\pi/2$ about any of the three axes x, y, and z, then the field-gradient tensor at the origin vanishes.

(6 points)

2.3.2.)

$$\boxed{(1-x^2)y'' - 2xy' + \lambda y = 0}$$

ansatz: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

(1)

$$\rightsquigarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + \lambda a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n(n+1)-\lambda)a_n\right] x^n \overset{!}{=} 0$$

(1)

$$\rightsquigarrow \underline{a_{n+2} = \frac{n(n+1)-\lambda}{(n+2)(n+1)} a_n} \qquad \text{recursion relation}$$

$\rightsquigarrow$ Necessary and sufficient for the series to terminate is

(i) $\lambda = n(n+1)$ with $n = 0, 1, \ldots$

(ii) $a_1 = 0$ for n = even

$a_0 = 0$ for n = odd

(1)

check: $n=0$: $a_0 = 1 \rightsquigarrow a_2 = 0 \rightsquigarrow a_{2n} = 0 \ \forall\, n > 0$

$a_1 = 0 \rightsquigarrow a_{2n+1} = 0 \ \forall\, n \geq 0$

$\underline{P_0(x) = 1}$ ✓

$n=1$: $a_0 = 0 \rightsquigarrow a_{2n} = 0 \ \forall\, n \geq 0$

$a_1 = 1 \rightsquigarrow a_3 = 0 \rightsquigarrow a_{2n+1} = 0 \ \forall\, n > 0$

$\underline{P_1(x) = x}$ ✓

$n=2$: $a_0 = \frac{-1}{2} \rightsquigarrow a_2 = \frac{-6}{2} a_0 = \frac{3}{2}$, $a_4 = 0$

$a_1 = 0$ $\rightsquigarrow \underline{P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}}$ ✓

(1)

p 2.3.3

2.3.3-) Start with the Legendre ODE

$$\boxed{(1-x^2)\frac{d^2}{dx^2}P_\ell(x) - 2x\frac{d}{dx}P_\ell(x) = -\ell(\ell+1)P_\ell(x)} \quad (0)$$

$$\frac{d}{dx}(0) \rightsquigarrow \quad (1-x^2)\frac{d^3}{dx^3}P_\ell(x) - 4x\frac{d^2}{dx^2}P_\ell(x) - 2\frac{d}{dx}P_\ell(x) = -\ell(\ell+1)\frac{d}{dx}P_\ell(x) \quad (1)$$

$$\frac{d}{dx}(1) \rightsquigarrow \quad (1-x^2)\frac{d^4}{dx^4}P_\ell(x) - 6x\frac{d^3}{dx^3}P_\ell(x) - 6\frac{d^2}{dx^2}P_\ell(x) = -\ell(\ell+1)\frac{d^2}{dx^2}P_\ell(x) \quad (2)$$

(m-1) times ⋮

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_\ell(x) - 2mx\frac{d^m}{dx^m}P_\ell(x) + \left[\ell(\ell+1) - a_{m-1}\right]\frac{d^{m-1}}{dx^{m-1}}P_\ell(x) = 0 \quad (m$$

(1)

where $a_n = a_{n-1} + 2n$

ansatz : $a_{n-1} = -n+n^2 \rightsquigarrow a_n = -(n+1)+(n+1)^2 = n+n^2 = a_{n-1}+2n$ ✓

$$\rightsquigarrow (1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_\ell(x) - 2mx\frac{d^m}{dx^m}P_\ell(x) + \underbrace{\left[\ell(\ell+1)+m-m^2\right]}_{=(\ell+m)(\ell-m+1)}\frac{d^{m-1}}{dx^{m-1}}P_\ell(x) = 0 \quad (*$$

(1)

Now we write

$$\sqrt{1-x^2}\frac{d}{dx}P_\ell^m(x) = \sqrt{1-x^2}(-)^m\frac{d}{dx}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

$$= \sqrt{1-x^2}(-)^m\frac{m}{2}(-2x)(1-x^2)^{\frac{m-2}{2}}\frac{d^m}{dx^m}P_\ell(x) + \sqrt{1-x^2}(-)^m(1-x^2)^{m/2}\frac{d^{m+1}}{dx^{m+1}}P_\ell$$

$$= -\frac{mx}{\sqrt{1-x^2}}P_\ell^m(x) + (-)^m(1-x^2)^{\frac{m-1}{2}}(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_\ell(x)$$

$$\overset{(m)}{=} \frac{-mx}{\sqrt{1-x^2}}P_\ell^m(x) + (-)^m(1-x^2)^{\frac{m-1}{2}}\left[2mx\frac{d^m}{dx^m}P_\ell(x) - (\ell+m)(\ell-m+1)\frac{d^{m-1}}{dx^{m-1}}P_\ell\right.$$

$$= \frac{-mx}{\sqrt{1-x^2}}P_\ell^m(x) + \frac{2mx}{\sqrt{1-x^2}}P_\ell^m(x) + (\ell+m)(\ell-m+1)P_\ell^{m-1}(x)$$

(1)

$$= \frac{mx}{\sqrt{1-x^2}}P_\ell^m(x) + (\ell+m)(\ell-m+1)P_\ell^{m-1}(x)$$

2.7.4) Start with the definition of the spherical harmonics

$$Y_\ell^m(\Omega) = N_\ell^m e^{im\varphi} P_\ell^m(\cos\vartheta)$$

with $$N_\ell^m = \left(\frac{(2\ell+1)(\ell-m)!}{4\pi\,(\ell+m)!}\right)^{1/2} \in \mathbb{R}\,, \quad P_\ell^m \in \mathbb{R} \text{ assoc. Legendre fcts}$$

(1) $$\underline{\underline{Y_\ell^m(\Omega)^*}} = N_\ell^m e^{-im\varphi} P_\ell^m(\cos\vartheta)$$

§2.2 remark (7)(iii)

$$= N_\ell^m e^{-im\varphi} \frac{(\ell+m)!}{(\ell-m)!} (-)^m P_\ell^{-m}(\cos\vartheta)$$

$$= \left(\frac{(2\ell+1)(\ell+m)!}{4\pi\,(\ell-m)!}\right)^{1/2} (-)^m e^{-im\varphi} P_\ell^{-m}(\cos\vartheta)$$

1

$$= (-)^m N_\ell^{-m} P_\ell^{-m}(\cos\vartheta) = \underline{\underline{(-)^m Y_\ell^{-m}(\cos\vartheta)}}$$

(2) $$\underline{\underline{\cos\vartheta\, Y_\ell^m(\Omega)}} = N_\ell^m e^{im\varphi} \eta P_\ell^m(\eta)$$

§2.2 remark (7)(v)

1

$$= N_\ell^m e^{im\varphi} \left(\frac{\ell+1-m}{2\ell+1} P_{\ell+1}^m(\eta) + \frac{\ell+m}{2\ell+1} P_{\ell-1}^m(\eta)\right)$$

$$= Y_{\ell+1}^m(\Omega)\, \frac{\ell+1-m}{2\ell+1}\, \frac{N_\ell^m}{N_{\ell+1}^m} + Y_{\ell-1}^m(\Omega)\, \frac{\ell+m}{2\ell+1}\, \frac{N_\ell^m}{N_{\ell-1}^m}$$

$$= \left(\frac{\cancel{2\ell+1}}{2\ell+3}\, \frac{\cancel{(\ell-m)!}}{\cancel{(\ell-m+1)!}}\, \frac{\cancel{(\ell+m+1)!}}{\cancel{(\ell+m)!}}\, \frac{(\ell+1-m)^{\cancel{2}}}{(2\ell+1)^{\cancel{2}}}\right)^{1/2} Y_{\ell+1}^m(\Omega)$$

$$+ \left(\frac{\cancel{2\ell+1}}{2\ell-1}\, \frac{\cancel{(\ell-m)!}}{\cancel{(\ell-m-1)!}}\, \frac{\cancel{(\ell+m-1)!}}{\cancel{(\ell+m)!}}\, \frac{\cancel{(\ell+m)}^2}{(2\ell+1)^{\cancel{2}}}\right)^{1/2} Y_{\ell-1}^m(\Omega)$$

1

$$= \underline{\underline{\left(\frac{(\ell+1+m)(\ell+1-m)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^m(\Omega) + \left(\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^m(\Omega)}}$$

(3) $\sin\vartheta\, e^{i\varphi}\, Y_\ell^m(\Omega) = \sqrt{1-\eta^2}\, e^{i\varphi} N_\ell^m e^{im\varphi} P_\ell^m(\eta)$

1 $\quad = N_\ell^m e^{i(m+1)\varphi} \sqrt{1-\eta^2}\, P_\ell^m(\eta)$

§7.2 remark(7)(vi)

$$= N_\ell^m e^{i(m+1)\varphi} \frac{1}{2\ell+1}\left[P_{\ell-1}^{m+1}(\eta) - P_{\ell+1}^{m+1}(\eta)\right]$$

$$= \frac{N_\ell^m}{N_{\ell-1}^{m+1}} \frac{1}{2\ell+1} Y_{\ell-1}^{m+1}(\Omega) - \frac{N_\ell^m}{N_{\ell+1}^{m+1}} \frac{1}{2\ell+1} Y_{\ell+1}^{m+1}(\Omega)$$

$$= \left(\frac{2\ell+1}{2\ell-1} \frac{(\ell-m)!}{(\ell-m-2)!} \frac{1}{(2\ell+1)^2}\right)^{1/2} Y_{\ell-1}^{m+1}(\Omega) - \left(\frac{2\ell+1}{2\ell+3} \frac{(\ell+m+2)!}{(\ell+m)!} \frac{1}{(2\ell+1)^2}\right) Y_{\ell+1}^{m+1}$$

1 $$= \left(\frac{(\ell-m)(\ell-m-1)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^{m+1}(\Omega) - \left(\frac{(\ell+m+2)(\ell+m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^{m+1}(\Omega) \quad (+$$

$$\sin\vartheta\, e^{-i\varphi}\, Y_\ell^m(\Omega) = \left(\sin\vartheta\, e^{i\varphi}\, Y_\ell^m(\Omega)^*\right)^* \overset{(1)}{=} (-)^m\left(\sin\vartheta\, e^{i\varphi}\, Y_\ell^{-m}(\Omega)\right)$$

$$\overset{(+)}{=} \left(\frac{(\ell+m)(\ell+m-1)}{(2\ell-1)(2\ell+1)}\right)^{1/2} (-)^m Y_{\ell-1}^{-m+1}(\Omega)^* - \left(\frac{(\ell-m+2)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} (-)^m Y_{\ell+1}^{-m+1}$$

1 $$\overset{(1)}{=} -\left(\frac{(\ell+m)(\ell+m-1)}{(2\ell-1)(2\ell+1)}\right)^{1/2} Y_{\ell-1}^{m-1}(\Omega) + \left(\frac{(\ell-m+2)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1}^{m-1}(\Omega$$

(4) $\hat{L}_- Y_\ell^m(\Omega) = N_\ell^m e^{-i\varphi}\left[-\frac{\partial}{\partial\vartheta} + i\cot\vartheta \frac{\partial}{\partial\varphi}\right] e^{im\varphi} P_\ell^m(\cos\vartheta)$

$\eta = \cos\vartheta \quad d\eta = -\sin\vartheta\, d\vartheta \quad d\vartheta = \frac{-1}{\sqrt{1-\eta^2}}\, d\eta$

1 $$= N_\ell^m e^{-i\varphi}\left[\sqrt{1-\eta^2}\frac{d}{d\eta} - m\frac{\eta}{\sqrt{1-\eta^2}}\right] e^{im\varphi} P_\ell^m(\eta)$$

$$= N_l^m e^{i(m-1)\varphi}\left[\sqrt{1-\eta^2}\,\frac{d}{d\eta} - m\,\frac{\eta}{\sqrt{1-\eta^2}}\right] P_l^m(\eta)$$

Problem 2.3.3

$$\overset{\Downarrow}{=} N_l^m e^{i(m-1)\varphi}\,(l+m)(l-m+1)\,P_l^{m-1}(\eta)$$

$$= \frac{N_l^m}{N_l^{m-1}}\,(l+m)(l-m+1)\,Y_l^{m-1}(\eta)$$

$$= \left(\frac{(l-m)!}{\cancel{(l-m+1)}!}\,\frac{(l+m-1)!}{(l+m)!}\,(l+m)^2\cancel{(l-m+1)^2}\right)^{1/2} Y_l^{m-1}(\Omega)$$

$$= \underline{\underline{\left((l-m+1)(l+m)\right)^{1/2}\, Y_l^{m-1}(\Omega)}} \qquad (+)$$

1

$$\underline{\underline{\hat{L}_+ Y_l^m(\Omega)}} = \left(-\hat{L}_- Y_l^m(\Omega)^*\right)^*$$

$$\overset{(1)}{=} \left((-)^{m+1}\,\hat{L}_- Y_l^{-m}(\Omega)\right)^*$$

$$\overset{(+)}{=} (-)^{m+1}\left((l+m+1)(l-m)\right)^{1/2} Y_l^{-m-1}(\Omega)^*$$

$$\overset{(1)}{=} \underline{\underline{\left((l+m+1)(l-m)\right)^{1/2}\, Y_l^{m+1}(\Omega)}}$$

1

2.3.5.) $\varphi(\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\varphi\Big|_{\vec{x}=0} + \frac{1}{2} x_i x_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}\Big|_{\vec{x}=0} + \dots$

$$=: \varphi_0 - \vec{x}\cdot\vec{E} + \frac{1}{2} x_i x_j \varphi_{ij} + \dots$$

$$\equiv \varphi_0 + \varphi_1(\vec{x}) + \varphi_2(\vec{x}) + \dots$$

a) $\rho(\vec{\xi}) = \rho(-\vec{\xi}) \rightsquigarrow \varphi(-\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}+\vec{\xi}|} = \int d\vec{\xi}\, \frac{\rho(-\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x})$

1 $\rightsquigarrow$ All terms odd in $\vec{x}$ vanish, in particular $\underline{\underline{\vec{E}=0}}$

b) φ_{ij} is real symmetric $\rightsquigarrow \exists$ coordinate system such that φ_{ij} is diagonal

1

$\varphi(\vec{x})$ obeys Laplace's eq $\forall\ |\vec{x}| < r_0$

$\rightsquigarrow \sum_i \varphi_{ii} = 0$

$\rightsquigarrow \varphi_{ij}$ has the form $\varphi_{ij} = \begin{pmatrix} \varphi_+ + \varphi_- & 0 & 0 \\ 0 & \varphi_+ - \varphi_- & 0 \\ 0 & 0 & -2\varphi_+ \end{pmatrix}$

1 where $\varphi_- = \frac{1}{2}(\varphi_{11} - \varphi_{22})$

$\rightsquigarrow \underline{\varphi_2(\vec{x})} = \frac{1}{2} r^2 \sin^2\vartheta \cos^2\varphi\ (\varphi_+ + \varphi_-)$

$+ \frac{1}{2} r^2 \sin^2\vartheta \sin^2\varphi\ (\varphi_+ - \varphi_-)$

$+ \frac{1}{2} r^2 \cos^2\vartheta\ (-2\varphi_+)$

1 $= \underline{\frac{1}{2} r^2 \left[(1 - 3\cos^2\vartheta)\varphi_+ + \sin^2\vartheta \cos 2\varphi\ \varphi_- \right]}$

Rotational invariance of $\rho(\vec{\xi})$ implies rotational invariance of $\varphi(\vec{x})$, and in particular of $\varphi_2(\vec{x})$

$$\rightsquigarrow \underline{\varphi_2(r,\vartheta,\varphi+\alpha)} = \frac{1}{2}r^2\left[(1-3\cos^2\vartheta)\,\varphi_+ + \sin^2\vartheta\cos 2(\varphi+\alpha)\,\varphi_-\right]$$

$$\stackrel{!}{=} \underline{\varphi_2(r,\vartheta,\varphi)}$$

1

$$\rightsquigarrow \varphi_-\cos(2\varphi+2\alpha) = \varphi_-\cos 2\varphi \quad \rightsquigarrow \underline{\underline{\varphi_- = 0}}$$

c) cubic symmetry $\rightsquigarrow$ $\rho(\vec{\xi})$ invariant under rotations through $\frac{\pi}{2}$ about any of the three axes x,y,z.

b) $\rightsquigarrow$ $\varphi_- = 0$ due to invariance under rotation about z-axis

Rotation about x or y $\rightsquigarrow$ $\vartheta \to \vartheta + \pi/2$

But invariance of $\rho(\vec{\xi})$ implies invariance of $\varphi(\vec{x})$

$$\rightsquigarrow \underline{\varphi_2(r,\vartheta+\tfrac{\pi}{2},\varphi)} = \frac{1}{2}r^2\varphi_+\left[1-3\cos^2(\vartheta+\pi/2)\right]$$

$$\stackrel{!}{=} \underline{\frac{1}{2}r^2\varphi_+\left[1-3\cos^2\vartheta\right]}$$

1

$$\rightsquigarrow \varphi_+\cos^2(\vartheta+\pi/2) = \varphi_+\cos^2\vartheta \quad \rightsquigarrow \underline{\varphi_+ = 0} \quad \rightsquigarrow \underline{\underline{\varphi_{ij} = 0}}$$