PHYS 610 Mathematical Methods for Scientists F 2023

Problem Assignment # 5 10/25/2023
due 11/01/2023

This homework assignment doubles as the Midterm. Treat it like any other homework assignment, but please do not collaborate on this one.

1.2.5 **Abelian groups**

Let $(G, \vee)$ be a group with neutral element e. Let $a \in G$ be a fixed element, and define a mapping $\varphi : G \to G$ by $\varphi(x) = a \vee x \vee a^{-1}\ \forall x \in G$.

a) Show that φ defines an automorphism on G, called an *inner automorphism*.

b) Show that abelian groups have no inner automorphisms except for the identity mapping $\varphi(x) = x$.

c) Let $g \vee g = e\ \forall g \in G$. Prove that G is abelian.

(6 points)

1.4.7. **Banach space**

Let B be a K-vector space (k $= \mathbb{R}$ or $\mathbb{C}$) with null vector θ. Let $||\ldots|| \ : \ \mathrm{B} \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0\ \forall\ x \in \mathrm{B}$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y||\ \forall\ x, y \in \mathrm{B}$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x||\ \forall\ x \in \mathrm{B}, \lambda \in \mathrm{K}$.

Then we call $||\ldots||$ a **norm** on B, and $||x||$ the **norm** of x.

Further define a mapping $d \ : \ \mathrm{B} \times \mathrm{B} \to \mathbb{R}$ by

$d(x, y) := ||x - y||\ \forall\ x, y \in \mathrm{B}$

Then we call $d(x, y)$ the **distance** between x and y.

a) Show that d is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space B with distance/metric d is complete, then we call B a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 example (3), and you don't have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let B^* be the dual space of B, i.e., the space of linear forms ℓ on B, and define a norm of ℓ by

$||\ell|| := \sup_{||x||=1} \{|\ell(x)|\}$

c) Show that the such defined norm on B^* is a norm in the sense of the norm defined on B above.

Note: In case you're wondering: B^* is complete, and hence a B-space, but the proof of completeness is difficult.

(8 points)