PHYS 610 Mathematical Methods for Scientists F 2023

Problem Assignment # 1 09/27/2023
due 10/04/2023

1.1.1 **Russell's Paradox (B. Russell, 1901)**

a) Consider the set M defined as the set of all sets that do not contain themselves as an element: $M = \{x; x \notin x\}$. Discuss why this is a problematic definition.

b) A less abstract version of Russell's paradox is known as the barber's paradox: Consider a town where all men either shave themselves, or let the barber shave them and don't shave themselves. Now consider the statement

The barber is a man in town who shaves all men who do not shave themselves, and only those.

Discuss why this definition of the barber is problematic (assuming there actually is a barber in town).

hint: Ask "Does the barber shave himself?"

c) Suppose the definition of the barber is modified to read

The barber shaves all men in town who do not shave themselves, and only those.

Discuss what this modification does to the paradox.

(3 points)

1.1.2 **Distributive property of the union and intersection relations**

Show graphically that the relations $\cup$ and $\cap$ defined in ch.1, §1.1, def. 3 obey the following distributive properties: For any three sets A, B, C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(2 points)

1.1.3 **Mappings**

Are the following $f : X \to Y$ true mappings? If so, are they surjective, or injective, or both?

a) $X = Y = \mathbb{Z}, \quad f(m) = m^2 + 1.$

b) $X = Y = \mathbb{N}, \quad f(n) = n + 1.$

c) $X = \mathbb{Z}, Y = \mathbb{R}, \quad f(x) = \log x.$

d) $X = Y = \mathbb{R}, \quad f(x) = e^x.$

(2 points)

1.1.4 **Parabolic Mapping**

Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = a\,n^2 + b\,n + c$, with $a, b, c \in \mathbb{Z}$.

a) For which triplets (a, b, c) is f surjective?

b) For which (a, b, c) is f injective?

(4 points)

1.1.1) a) Suppose R contains itself as an element. Then, by its definition, it does not.

Suppose R does not contain itself as an element. Then, again by its definition, it does.

There is no third possibility, so the definition is

1 logically self-contradictory.

b) Suppose the barber shaves himself. Since he is the barber, this means that he is shaved by the barber and by definition does not shave himself.

Suppose the barber does not shave himself. Then he is shaved by the barber, and then he does shave himself.

1 This is the same logical problem as in part a).

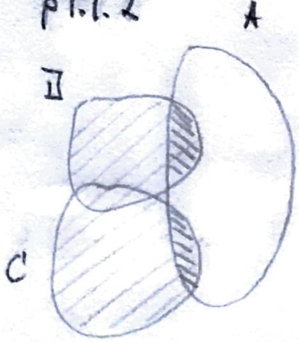
c) By dropping the requirement that the barber "is a man in town" the problem goes away: the barber can be a woman, or a man from a different town.

NB: A logically possible conclusion from a) is also

1 that the town has no barber.

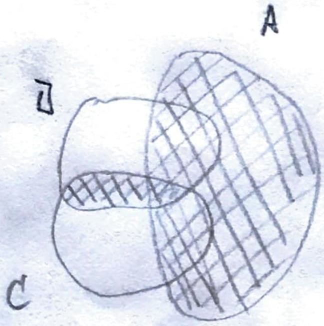
p1.1.2

1.1.2.) $A \cap (B \cup C) =$



$= (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) =$



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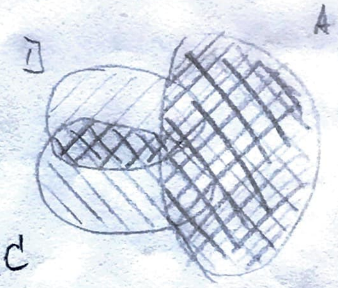
$(A \cup B) \cap (A \cup C) =$



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1.1.3.) a) f is a mapping. It is neither surjective ($f(n) \geq 1 \; \forall n \in \mathbb{Z}$)

1/2 nor injective ($f(-n) = f(n) \; \forall n \in \mathbb{Z}$).

b) f is a mapping. It is not surjective ($1 \in \mathbb{N}$ has no

1/2 pre-image). It is injective, since $n+1$ is monotonic.

1/2 c) f is not a mapping, since $x \leq 0$ have no image.

d) f is a mapping, since e^x is defined $\forall x \in \mathbb{R}$. It is

not surjective, since $f(x) > 0 \; \forall x \in \mathbb{R}$. It is injective,

1/2 since e^x is monotonic.

1.1.4.) a) $f(n)$ has an absolute minimum if $a \neq 0$

$\rightarrow$ $a = 0$ is necessary for f to be surjective.

Now consider $f(n) = bn + c$

If $b = 0$, then $f(n) \equiv c$ and hence not surjective.

If $b \geq 2$ or $b \leq -2$, then $f(n)$ never equals $c \pm 1$, and hence f is not surjective.

If $b = \pm 1$ for any c, then $f(n)$ covers all of $\mathbb{Z}$.

$\rightarrow$ f is surjective for $(a, b, c) = (0, \pm 1, c \in \mathbb{Z})$

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b) For f to be injective, $f(n) = f(m)$ must imply $n = m$.

Let $n = m + x$, with $x \in \mathbb{Z}$.

Then $f(n) = f(m)$ takes the form

$$\cancel{am^2} + bm = \cancel{am^2} + 2axm + ax^2 \cancel{+ bm} + bx$$

$$\rightarrow \quad ax^2 + (2am + b)x = 0 \qquad (*)$$

$x = 0$ is always a solution, which implies $n = m$.

For $x \neq 0$, the only solution of $(*)$ is

$$x = -2m - b/a$$

As long as this solution is $\notin \mathbb{Z}$, f is injective.

$\rightarrow$ For f to be injective, b must not be divisible by a in $\mathbb{Z}$.

$\rightarrow$ f is injective for $(a \in \mathbb{Z}, b \in \mathbb{Z} \setminus a\mathbb{Z}, c \in \mathbb{Z})$

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