PHYS 610 Mathematical Methods for Scientists F 2023

Problem Assignment # 3 10/11/2023
due 10/18/2023

1.2.3 **The group** S_3

a) Compile the group table for the symmetric group S_3. Is S_3 abelian?

b) Find all subgroups of S_3. Which of these are abelian?

(6 points)

1.2.4 **Subgroups**

Let $(G, \vee)$ be a group and let $H \subset G$ with $H \neq \emptyset$. Show that H is a subgroup of G if and only if $a, b \in H$ implies $a \vee b^{-1} \in H$.

(5 points)

1.3.1 **Fields**

a) Show that the set of rational numbers $\mathbb{Q}$ forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set F with two elements, $F = \{\theta, e\}$. On F, define an operation "plus" $(+)$, about which we assume nothing but the defining properties

$$\theta + \theta = \theta \quad , \quad \theta + e = e + \theta = e \quad , \quad e + e = \theta$$

Further, define a second operation "times" $(\cdot)$, about which we assume nothing but the defining properties

$$\theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta \quad , \quad e \cdot e = e$$

Show that with these definitions (and **no** additional assumptions), F is a field.

(7 points)

1.4.1. **Function space**

Consider the set C of continuous functions $f : [0, 1] \to \mathbb{R}$. Show that by suitably defining an addition on C, and a multiplication with real numbers, one can make C an additive vector space over $\mathbb{R}$.

(2 points)

1.2.3-10) The elements of S_3 are

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

1 $$P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

With this representation, the group table is

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_5	P_6	P_3	P_4
P_3	P_3	P_4	P_1	P_2	P_6	P_5
P_4	P_4	P_3	P_6	P_5	P_1	P_2
P_5	P_5	P_6	P_2	P_1	P_4	P_3
P_6	P_6	P_5	P_4	P_3	P_2	P_1

1 S_3 is not abelian: E.g., $P_2 \circ P_3 = P_5$, $P_3 \circ P_2 = P_4$.

b) Consider the group table from problem a). Now consider subsets of S_3 that contain

5 elements: $\{P_2, P_3, P_4, P_5, P_6\}$ does not contain $P_1 = E$

$\{P_1, P_3, P_4, P_5, P_6\}$ is not closed, since $P_3 \circ P_4 = P_2$

same for the other 4 possibilities

1

4 elements: The subset must contain P_1 -> We can form

$\{P_1, P_2, P_3, P_4\}$ not closed since $P_2 \circ P_3 = P_5$

$\{P_1, P_2, P_3, P_5\}$ " since $P_3 \circ P_2 = P_4$

$\{P_1, P_2, P_4, P_5\}$ " since $P_4 \circ P_2 = P_3$

$\{P_1, P_3, P_4, P_5\}$ " since $P_3 \circ P_4 = P_2$

same for the other 6 possibilities

1

3 elements: Consider $\{P_1, P_4, P_5\}$, which has a group table

	P_1	P_4	P_5
P_1	P_1	P_4	P_5
P_4	P_4	P_5	P_1
P_5	P_5	P_1	P_4

This is an abelian subgroup!

Whereas,

$\{P_1, P_2, P_3\}$ is not closed since $P_2 \circ P_3 = P_5$

and the same for the other 8 possibilities.

1

2 elmts : $\{P_1, P_2\}$ is an abelian subgroup

$\{P_1, P_3\}$ "

$\{P_1, P_4\}$ is not closed

$\{P_1, P_5\}$ "

$\{P_1, P_6\}$ is an abelian subgroup

(1)

1 elmt : $\{P_1\}$ trivially is an abelian subgroup

→ The subgroups of S_3 are:

$S_3^{(1)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$
$S_3^{(2)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$
$S_3^{(3)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\}$
$S_3^{(4)} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$

They are all abelian

1.2.4 Subgroups

(1) Show that $(a, b \in H \rightsquigarrow a \vee b^{-1} \in H) \rightsquigarrow H$ is a subgroup

Suppose $a, b \in H \rightsquigarrow a \vee b^{-1} \in H$

In particular, $b = a \in H \rightsquigarrow a \vee a^{-1} = e \in H$

and if $a = e$, then $e \vee b^{-1} = b^{-1} \in H$

$\rightsquigarrow$ Axioms (iii), (iv) from §2.1 are fulfilled

Axiom (ii) is fulfilled since G and H share the associative $\vee$

Now consider $a \vee b = a \vee (b^{-1})^{-1} \in H$ since $b^{-1} \in H$ if $b \in H$

$\rightsquigarrow$ Axiom (i) is fulfilled

$\rightsquigarrow$ H is a group $\rightsquigarrow$ The condition is sufficient

(2) Show that $(a, b \in H$ does not imply $a \vee b^{-1} \in H) \rightsquigarrow H$ is not a subgroup

Suppose $\exists\, a, b \in H : a \vee b^{-1} \notin H$

In order for H to be a group, $b \in H$ must imply $b^{-1} \in H$

Now we have $a, b^{-1} \in H$, but $a \vee b^{-1} \notin H$

$\rightsquigarrow$ Axiom (i) is violated $\rightsquigarrow$ H is not a group

$\rightsquigarrow$ The condition is necessary

1.3.1.) a) $\mathbb{Q}$ is a group under addition with neutral element $0 \in \mathbb{Q}$:

(i) $q_1 + q_2 \in \mathbb{Q} \;\; \forall\, q_1, q_2 \in \mathbb{Q}$

(ii) Addition is associative and commutative

(iii) The number zero is an element of $\mathbb{Q}$, and $0 + q = q \;\; \forall q \in \mathbb{Q}$

1 (iv) Let $q \in \mathbb{Q}$. Then $\exists\, -q : \; q + (-q) = 0$

$\mathbb{Q} \setminus \{0\}$ is also a group under multiplication:

(i) $q_1 q_2 \in \mathbb{Q} \;\; \forall\, q_1, q_2 \in \mathbb{Q}$

(ii) Multiplication is associative and commutative

(iii) The number 1 is an element of $\mathbb{Q}$, and $1 \cdot q = q \;\; \forall\, q \in \mathbb{Q}$

(iv) Let $q \in \mathbb{Q}$ and $q \neq 0$. Then $\exists\; q^{-1} \equiv \frac{1}{q} : \; q q^{-1} = 1$.

1 Finally, ordinary addition and multiplication on $\mathbb{Q}$ are distributive.

$\rightarrow$ $\mathbb{Q}$ is a commutative field b.) We need to show that F is a group under addition.

(i) $a+b \in F \ \forall a,b \in F$ by definition $\Rightarrow$ closure ✓

(ii) $(d+e)+d = e+d = e = d+(e+d)$

$(e+e)+d = d+d = d = e+(e+d)$

1 $\Rightarrow$ "+" is associative

(iii) d is the neutral element by definition.

(iv) $-d=d$, $-e=e$ by definition $\Rightarrow$ existence of inverse ✓

(v) "+" is commutative by definition

1 $\Rightarrow$ F is an abelian group under "+".

We also need to show that $F\setminus\{d\}$ is a group under "$\cdot$".

But $F\setminus\{d\} = \{e\}$, and

(i) closure ✓ by definition

(ii) associativity is trivial

(iii) e is neutral element by definition

(iv) e is its own inverse

1 $\Rightarrow$ $F\setminus\{d\}$ is a group under "$\cdot$". It is trivially abelian.

Finally, we must check the distribution laws. Since "$\cdot$" is abelian we only have to show that $(a+b)\cdot c = a\cdot c + b\cdot c \ \forall a,b,c \in F$. (*)

1 (i) $c=d$. $\Rightarrow (a+b)\cdot d = d = a\cdot d + b\cdot d$ irrespective of a,b ✓

(ii) $c=e$. If either $a=d$ or $b=d$, (*) holds.

If $a=b=e$, $(e+e)\cdot e = d\cdot e = d$

1 and $e\cdot e + e\cdot e = d+d = d$ $\Rightarrow$ distribution law ✓

$\Rightarrow$ F is a field

1.4.1.) On C, define $(f+g)(x) := f(x)+g(x)$

If f and g are continuous, then so is the well defined $(f+g)$

$\rightsquigarrow$ closure ✓

Furthermore, since $f(x) \in \mathbb{R}$, C inherits all of the other group properties from $(\mathbb{R},+)$

1 $\rightsquigarrow$ C is an additive group

Now define multiplication with scalars $\lambda \in \mathbb{R}$ by $(\lambda f)(x) := \lambda f(x)$

If f is continuous, then so is the well defined (λf).

Furthermore, since $\lambda \in \mathbb{R}$ and $f(x) \in \mathbb{R}$, this multiplication with scalars is bilinear and associative, as it inherits these properties from $\mathbb{R}$ under ordinary addition and multiplication of numbers

Finally, $(1f)(x) = 1f(x) = f(x) \; \forall x \in [0,1] \rightsquigarrow 1f = f$

1 $\rightsquigarrow$ C is an $\mathbb{R}$-vector space $\square$