PHYS 610 Mathematical Methods for Scientists F 2023

Problem Assignment # 4 10/18/2023
due 10/25/2023

1.4.2. **The space of rank-2 tensors**

a) Prove the theorem of ch.1 §4.3: Let V be a vector space V of dimension n over K. Then the space of rank-2 tensors, defined via bilinear forms $f : V \times V \to K$, forms a vector space of dimension n^2.

b) Consider the space of bilinear forms f on V that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

hint: On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

1.4.3. **Cross product of 3-vectors**

Let $x, y \in \mathbb{R}_3$ be vectors, and let ϵ_{ijk} be the Levi-Civita symbol. Show that the (covariant) components of the cross product $x \times y$ are given by

$$(x \times y)_i = \epsilon_{ijk} x^j y^k$$

(1 point)

1.4.5. $\mathbb{R}$ **as a metric space**

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

1.4.6. **Limits of sequences**

a) Show that a sequence in a metric space has at most one limit.

hint: Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequency with a limit is a Cauchy sequence.

(3 points)

1.4.2 (a) We know that the rank-2 tensors are one-to-one correspondent to bilinear forms $f(x,y)$. On the set of bilinear forms, define an addition by

$$(f+g)(x,y) := f(x,y) + g(x,y)$$

This makes the set of forms an additive group. Define a multiplication with scalars by

$$(\lambda f)(x,y) := \lambda f(x,y) \quad , \quad \lambda \in k$$

This makes the space of forms a k-vector space. On the space of rank-2 tensors $t, u, \ldots$ this corresponds to defining the tensor $t+u$ as the tensor with coordinates

1

$$(t+u)_{ij} = t_{ij} + u_{ij}$$

and the tensor λt as the one with coordinates

$$(\lambda t)_{ij} = \lambda t_{ij}$$

1 The space of tensors is now a k-vector space

Consider a basis $\{e_i\}$ of V, and construct n^2 tensors

$$E_{ij} := e_i \otimes e_j$$

with (contravariant) coordinates

$$(E_{ij})^{kl} = \delta_i^{\,k} \delta_j^{\,l}$$

Define a tensor t as a linear combination of the E_{ij},

$$t = \sum_{ij} \tau^{ij} E_{ij} \quad \text{with coefficients } \tau^{ij} \in k$$

This tensor has coordinates

$$t^{kl} = \sum_{ij} \tau^{ij} (E_{ij})^{kl} = \tau^{kl}$$

→ Any rank-2 tensor can be written as a linear combination of the E_{ij}, with the coordinates t^{ij} of t as the coefficients:

$$t = \sum_{ij} t^{ij} E_{ij}$$

1 → The E_{ij} span the space

But in order for t to be the null tensor, all of its coordinates must be zero, so $t = 0$ implies $t^{ij} = 0 \; \forall i,j$

→ The E_{ij} are linearly independent

→ The n^2 rank-2 tensors E_{ij} form a basis of the space of rank-2 tensors, and hence the space has dimension n^2.

1

b) Let f_{ij} be the bilinear form that corresponds to the tensor E_{ij}. Then

$$f_{ij}(e_k, e_l) = (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \quad \text{with } \delta_{ik} \text{ the Kronecker symbol}$$

For arbitrary $x, y \in V$ we have

$$f_{ij}(x,y) = x^k y^l f(e_k, e_l) = x^k y^l \delta_{ik} \delta_{jl} = x^i y^j$$

→ The set of n^2 bilinear forms f_{ij} defined by

$$\boxed{f_{ij}(x,y) = x^i y^j}$$

1 forms a basis of the space of bilinear forms.

1.4.3. **Cross product of 3-vectors**

Let $x, y \in \mathbb{R}_3$ be vectors, and let ϵ_{ijk} be the Levi-Civita symbol. Show that the (covariant) components of the cross product $x \times y$ are given by

$$(x \times y)_i = \epsilon_{ijk} x^j y^k$$

(1 point)

Solution

Calculate

$$\epsilon_{ijk} x^j y^k = \begin{cases} x^2 y^3 - x^3 y^2 & \text{if} \quad i = 1 \\ x^3 y^1 - x^1 y^3 & \text{if} \quad i = 2 \\ x^1 y^2 - x^2 y^1 & \text{if} \quad i = 3 \end{cases}$$

This is consistent with the usual definition of the cross product, provided we interpret $x^2 y^3 - x^3 y^2 = (x \times y)_1$ and cyclic as the **covariant** components of $x \times y$.

(1) 1.4.5.) Positive definiteness and symmetry are obvious.

Prove the triangle inequality:

By definition of $|x|$, we have $xy \leq |x| \cdot |y| \; \forall x, y \in \mathbb{R}$

$\rightsquigarrow \quad 0 \leq 2(x-y)(z-y) + 2|x-y| \cdot |z-y|$

(1)

$\rightsquigarrow \quad (x-z)^2 = x^2 - 2xz + z^2 \leq x^2 - 2xz + z^2 + 2(x-y)(z-y) + 2|x-y| \cdot |z-y|$

$= x^2 - 2xz + z^2 + 2(x-y)z - 2(x-y)y + 2|x-y| \cdot |z-y|$

$= x^2 - \cancel{2xz} + z^2 + \cancel{2xz} - 2xy + 2y^2 - 2yz + 2|x-y| \cdot |z-y|$

$= x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + 2|x-y| \cdot |z-y|$

$= (x-y)^2 + (y-z)^2 + 2|x-y| \cdot |y-z|$

$= (|x-y| + |y-z|)^2$

But $(x-z)^2 \geq 0 \rightsquigarrow$

$|x-z| \leq |x-y| + |y-z|$ triangle inequality

(1)

$\square$

1.4.6.) a) Let x_n be a sequence. Suppose $x_n \Rightarrow x^*$ and $x_n \Rightarrow y^*$.

$\rightsquigarrow \rho(x^*, y^*) \leq \rho(x^*, x_n) + \rho(y^*, x_n) \quad \forall x_n$ by the triangle ineq.

But $\lim\limits_{n\to\infty} \rho(x^*, x_n) = \lim\limits_{n\to\infty} \rho(y^*, x_n) = 0$

1 $\quad \rightsquigarrow \rho(x^*, y^*) = 0 \quad \rightsquigarrow \quad x^* = y^*$ $\square$

b) Let x_n have a limit x^* : $x_n \Rightarrow x^*$

1 $\quad \rightsquigarrow \rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*)$

Let $\delta > 0$. Then $\exists N \in \mathbb{N}$: $\rho(x_n, x^*) < \delta \quad \forall n > N$

Now let $\varepsilon > 0$ and $\delta = \varepsilon/2$. Then $\exists N > 0$:

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

1 $\quad$ provided $n, m > N$. $\square$

1.4.7) a) $d(x-y) = \|x-y\| \geq 0 \quad \forall x,y \in \mathcal{B}$ by property (i) of $\|\ldots\|$

and $d(x-y) = 0$ iff $x-y = \mathcal{O} \Leftrightarrow x = y$

$\rightsquigarrow$ positive definiteness ✓

$d(y-x) = \|y-x\| = \|-(x-y)\| = \|x-y\|$ by property (iii)

$= d(x-y)$

$\rightsquigarrow$ symmetry ✓

$d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\|$ by property (ii)

$= d(x,y) + d(y,z)$

(1) $\rightsquigarrow$ triangle inequality ✓

b) Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space and define

$\|x\| := |x| \quad \forall x \in \mathbb{R}$

Then $\|\ldots\| : \mathbb{R} \to \mathbb{R}$ has all of the properties required of a norm. Furthermore, §4.5 ex (3) $\rightsquigarrow$ every Cauchy sequence has a limit $\rightsquigarrow$ $\mathbb{R}$ is complete and hence a $\mathcal{B}$-space.

Same for $\mathbb{C}$ with a norm defined by

$\|z\| := |z| = \sqrt{z_1^2 + z_2^2}$

(1) This makes $\mathbb{C}$ a $\mathcal{B}$-space (assuming completeness)

1 c) Check the requirements from the definition of a norm:

(i) $\|l\| = \sup_{\|x\|=1} \{|l(x)|\} \Rightarrow \|l\| \geq 0$ since $|l(x)| \geq 0$

1/2

The null vector in $\mathcal{B}^*$ is the null functional l_0 defined by $l_0(x) = 0 \ \forall x \in \mathcal{B}$.

$\Rightarrow \|l_0\| = 0$

Conversely, let $\|l\| = 0$. Since the set of $x \in \mathcal{B}$ with $\|x\| = 1$ spans $\mathcal{B}$, l must equal l_0

1 $\Rightarrow \|l\| = 0$ iff $l = l_0$

(ii) $\|l_1 + l_2\| = \sup_{\|x\|=1} \{|l_1(x) + l_2(x)|\} \leq \sup \{|l_1(x)| + |l_2(x)|\}$

$= \|l_1\| + \|l_2\|$

1 That is, $\mathcal{B}^*$ inherits the triangle inequality from $\mathbb{C}$.

(iii) $\|a l\| = \sup_{\|x\|=1} \{|a l(x)|\} = \sup \{|a| \cdot |l(x)|\} = |a| \cdot \sup \{l(x)\}$

1/2 $= |a| \cdot \|l\| \ \forall a \in \mathbb{C}, l \in \mathcal{B}^*$

$\square$