Problem Assignment # 6 11/01/2023
due 11/08/2023

1.4.9. **Lorentz transformations in** M_2

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

1.4.11. **Special Lorentz transformations in** M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu$ (rotations)

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu$ (Lorentz boost along the x-direction)

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu$ (parity)

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu$ (time reversal)

. . . **/over**

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

1.5.1. **Transformations of tensor fields**

a) Consider a covariant rank-n tensor field $t_{i_1 \ldots i_n}(x)$ and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how $\tilde{t}_{i_1 \ldots i_n}(\tilde{x})$ is related to $t_{i_1 \ldots i_n}(x)$.

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector x_i (the case $n = 1$), and (ii) the covariant components of the metric tensor g_{ij}.

(4 points)

1.5.2. **Curl and divergence**

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

1.5.3. **Tensor products, and tensor traces**

Prove Propositions 1 and 2 from ch. 1 §5.3.

(4 points)

1.4.9, a) Let $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For Λ to be a Lorentz transfo, we must have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2-b^2 & ac-bd \\ ac-bd & c^2-d^2 \end{pmatrix}$$

From this obtain three constraints on the four numbers a,b,c,d

(i) $a^2-b^2=1$

(ii) $c^2-d^2=-1$

(1) (iii) $ac-bd=0$

Now consider $\sinh\phi$, which maps $\mathbb{R}$ one-to-one onto itself

$\rightsquigarrow \forall\, b\in\mathbb{R}\ \exists!\, \phi\in\mathbb{R}:\quad \underline{b=\sinh\phi}$

(i) $\rightsquigarrow\ a^2=1+b^2=1+\sinh^2\phi=\cosh^2\phi\ \rightsquigarrow\ \underline{a=\sigma\cosh\phi},\ \underline{\sigma=\pm1}$

Analogously, $\underline{c=\sinh\psi}$

(1) (ii) $\rightsquigarrow\ d^2=1+c^2=1+\sinh^2\psi=\cosh^2\psi\ \rightsquigarrow\ \underline{d=\tau\cosh\psi},\ \underline{\tau=\pm1}$

Finally,

(iii) $\rightsquigarrow\ 0=\sigma\cosh\phi\sinh\psi-\tau\cosh\psi\sinh\phi$

$=\cosh\phi\sinh\psi-\sigma\tau\cosh\psi\sinh\phi$

$=\cosh(\sigma\tau\phi)\sinh\psi-\cosh\psi\sinh(\sigma\tau\phi)$

(1) $=\sinh(\psi-\sigma\tau\phi)\ \rightsquigarrow\ \underline{\psi=\sigma\tau\phi}$

$$\rightsquigarrow \underline{\Lambda_{\sigma,\tau}(\phi)} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sinh(\sigma\tau\phi) & \tau\cosh(\sigma\tau\phi) \end{pmatrix} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\tau\sinh\phi & \tau\cosh\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\Lambda(\phi)\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}} \quad \text{with} \quad \underline{\Lambda(\phi)=\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix},\ \phi\in\mathbb{R}}$$

(1) and $\sigma,\tau=\pm1$ is the most general element of $O(1,1)$.

b) (i) $\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix}\begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$

$\rightarrow$ $D(\phi_1)D(\phi_2) = D(\phi_1+\phi_2)$ closure ✓

(ii) Matrix multiplication is associative ✓

(iii) $D(\phi=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$ neutral element ✓

(iv) $D(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$

and $\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\rightarrow$ $D(-\phi) = (D(\phi))^{-1}$ inverse ✓

$\rightarrow$ $\{D(\phi)\}$ is a group $SO^+(1,1)$

1 (i), (iii), (iv) provide the isomorphism $SO^+(1,1) \cong R(+)$

c) $e^{J\phi} = \mathbb{1}_2 + J\phi + \frac{1}{2}J^2\phi^2 + \dots$

and $\cosh\phi = 1 + \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 + \dots$, $\sinh\phi = \phi + \frac{1}{3!}\phi^3 + \dots$

Try $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\rightarrow$ $J^2 = \mathbb{1}_2$, $J^3 = J$, etc.

$\rightarrow$ $e^{J\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\phi^2 & 0 \\ 0 & \frac{1}{2}\phi^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}\phi^3 \\ \frac{1}{3!}\phi^3 & 0 \end{pmatrix} + \dots$

$= \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$ ✓

1 $\rightarrow$ J is the generator of $SO^+(1,1)$.

1.4.11 (?)10) i) $\underline{R^\mu{}_\alpha g_{\mu\nu} R^\nu{}_\lambda} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^T \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^T \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -R \end{array}\right)$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -R^T R \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right) = \underline{\underline{g_{\alpha\lambda}}} \quad \checkmark$$

ii) $\underline{\Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\lambda} = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\times \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^2\alpha - \sinh^2\alpha & 0 & 0 & \cosh\alpha\sinh\alpha - \sinh\alpha\cosh\alpha \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sinh\alpha\cosh\alpha - \cosh\alpha\sinh\alpha & 0 & 0 & \sinh^2\alpha - \cosh^2\alpha \end{pmatrix}$$

(1)

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \underline{\underline{g_{\alpha\lambda}}} \quad \checkmark$$

iii) $\underline{P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \underline{\underline{g_{\alpha\lambda}}} \quad \checkmark$

(1) iv) $\underline{T^\mu{}_\alpha g_{\mu\nu} T^\nu{}_\lambda} = (-)^2 P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda = \underline{\underline{g_{\alpha\lambda}}} \quad \checkmark$

(1) b) $R^\mu{}_\nu$ leaves the time coordinate invariant, and we know that the $R^i{}_j$ form a group (see PHYS 611) $\to$ $\{R^i{}_j\}$ is a subgroup of L.
The Lorentz boosts $\Lambda^\mu{}_\nu$ are a subgroup of L according to Problem 1.8.

(1) c) group tables:

	$\mathbb{1}$	P
$\mathbb{1}$	$\mathbb{1}$	P
P	P	$\mathbb{1}$

and

	$\mathbb{1}$	T
$\mathbb{1}$	$\mathbb{1}$	T
T	T	$\mathbb{1}$

$\to$ $\{\mathbb{1}, P\}$, and $\{\mathbb{1}, T\}$ are subgroups of L under matrix multiplication

	$\mathbb{1}$	P	T	PT
$\mathbb{1}$	$\mathbb{1}$	P	T	PT
P	P	$\mathbb{1}$	PT	T
T	T	PT	$\mathbb{1}$	P
PT	PT	T	P	$\mathbb{1}$

$\to$ $\{\mathbb{1}, P, T, PT\}$ is also a subgroup of L

1.5.1.) a) There are various ways to do this. One option is to start with the transformation property of contravariant tensor fields, §5.1, and use the metric tensor to lower the indices:

$$\underline{\underline{\tilde{t}_{i_1\dots i_N}(\tilde{x})}} = \tilde{g}_{i_1 j_1}\cdots\tilde{g}_{i_N j_N}\,\tilde{t}^{\,j_1\dots j_N}(\tilde{x})$$

$$\overset{\S 5.1(x)+\tilde{g}=g}{=} g_{i_1 j_1}\cdots g_{i_N j_N}\,D^{j_1}{}_{\lambda_1}\cdots D^{j_N}{}_{\lambda_N}\,t^{\lambda_1\dots\lambda_N}(x)$$

$$= g_{i_1 j_1}\cdots g_{i_N j_N}\,D^{j_1}{}_{\lambda_1}\cdots D^{j_N}{}_{\lambda_N}\,g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\,t_{l_1\dots l_N}(x)$$

$$= (gD)_{i_1\lambda_1}\cdots(gD)_{i_N\lambda_N}\,g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\,t_{l_1\dots l_N}(x)$$

$$\overset{gD=D^{T^{-1}}g}{=} \left((D^T)^{-1}g\right)_{i_1\lambda_1}\cdots\left((D^T)^{-1}g\right)_{i_N\lambda_N}\,g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\,t_{l_1\dots l_N}(x)$$

$$= \left((D^T)^{-1}\right)_{i_1}{}^{m_1} g_{m_1\lambda_1}\cdots\left((D^T)^{-1}\right)_{i_N}{}^{m_N} g_{m_N\lambda_N}\,g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\,t_{l_1\dots l_N}$$

$$= \left((D^T)^{-1}\right)_{i_1}{}^{m_1}\underbrace{g_{m_1\lambda_1}g^{\lambda_1 l_1}}_{=\delta_{m_1}{}^{l_1}}\cdots\left((D^T)^{-1}\right)_{i_N}{}^{m_N}\underbrace{g_{m_N\lambda_N}g^{\lambda_N l_N}}_{=\delta_{m_N}{}^{l_N}}\,t_{l_1\dots l_N}(x)$$

(2) $$\underline{\underline{= \left((D^T)^{-1}\right)_{i_1}{}^{j_1}\cdots\left((D^T)^{-1}\right)_{i_N}{}^{j_N}\,t_{j_1\dots j_N}(x)}}$$

This is §5.1 (x) with $D^{i_n}{}_{j_n}$ replaced by $\left((D^T)^{-1}\right)_{i_n}{}^{j_n}$ $(n=1,\dots,N)$

b) Special case $n=1$:

(1) $$\underline{\underline{\tilde{x}_i}} = \tilde{g}_{ij}\tilde{x}^j \overset{\S 4.8.2}{=} \tilde{g}_{ij}D^j{}_\lambda x^\lambda \overset{\tilde{g}=g}{=} (gD)_{i\lambda}x^\lambda \overset{gD=D^{T^{-1}}g}{=} \left((D^T)^{-1}g\right)_{i\lambda}x^\lambda$$
$$= \left((D^T)^{-1}\right)_i{}^j g_{j\lambda}x^\lambda = \underline{\underline{\left((D^T)^{-1}\right)_i{}^j}}\,x_j \quad ✓$$

(1) $$\tilde{g}_{ij} = g_{ik}g_{jl}\tilde{g}^{kl} \overset{\S 5.1}{=} g_{ik}g_{jl}D^k{}_m D^l{}_n g^{mn} = (gD)_{im}(gD)_{jn}g^{mo}g^{np}g_{op}$$
$$= (D^{T^{-1}}g)_{im}\,g^{mo}\,(D^{T^{-1}}g)_{jn}\,g^{np}\,g_{op} = \left((D^T)^{-1}\right)_i{}^o\left((D^T)^{-1}\right)_j{}^p\,g_{op} \quad ✓$$

1.5.2.) Consider the curl as defined in §5.2:

$$c^i(x) = \varepsilon^{ijl}\,\partial_j v_l(x)$$

$$\rightarrow \underline{\tilde{c}^i(\tilde{x})} = \tilde{\varepsilon}^{ijl}\,\tilde{\partial}_j \tilde{v}_l(\tilde{x})$$

Prob 25

$$\overset{\downarrow}{=} \tilde{\varepsilon}^{ijl}(D^{-1})^e{}_j\,\partial_e (D^{-1})^m{}_l\, v_m(x)$$

$$= \delta^i{}_n\,\tilde{\varepsilon}^{njl}(D^{-1})^e{}_j (D^{-1})^m{}_l\,\partial_e v_m(x)$$

$$= D^i{}_p (D^{-1})^p{}_n\,\tilde{\varepsilon}^{njl}(D^{-1})^e{}_j (D^{-1})^m{}_l\,\partial_e v_m(x)$$

$$= D^i{}_p\,\underbrace{(D^{-1})^p{}_n (D^{-1})^e{}_j (D^{-1})^m{}_l\,\tilde{\varepsilon}^{njl}}_{=(\det D)\,\varepsilon^{pem} \text{ by §5.1 remark (9)}}\,\partial_e v_m(x)$$

$$= (\det D)\, D^i{}_p\,\underbrace{\varepsilon^{plm}\,\partial_l v_m(x)}_{=c^p(x)}$$

(2)

$$= \underline{(\det D)\, D^i{}_p\, c^p(x)}$$

$\rightarrow$ $c^i(x)$ transforms as a pseudovector field.

Now the divergence: $d(x) = \partial_i v^i(x)$

$$\rightarrow \underline{\tilde{d}(\tilde{x})} = \tilde{\partial}_i \tilde{v}^i(\tilde{x}) = ((D^T)^{-1})_i{}^j\,\partial_j D^i{}_l v^l(x)$$

$$= \underbrace{(D^T)_l{}^i ((D^T)^{-1})_i{}^j}_{=\delta_l{}^j}\,\partial_j v^l(x)$$

(1)

$$= \underline{\partial_l v^l(x)}$$ $\rightarrow$ $d(x)$ transforms as a scalar field

1.5.2) Consider the tensor product

$$u^{i_1 \dots i_{N+M}} = s^{i_1 \dots i_N} \, t^{i_{N+1} \dots i_{N+M}}$$

Let both s and t be tensors. Then

$$\underline{\underline{\tilde{u}^{i_1 \dots i_{N+M}}}} = \tilde{s}^{i_1 \dots i_N} \, \tilde{t}^{i_{N+1} \dots i_{N+M}}$$

$$= D^{i_1}{}_{j_1} \cdots D^{i_N}{}_{j_N} \, s^{j_1 \dots j_N} \, D^{i_{N+1}}{}_{j_{N+1}} \cdots D^{i_{N+M}}{}_{j_{N+M}} \, t^{j_{N+1} \dots j_{N+M}}$$

$$= \underline{\underline{D^{i_1}{}_{j_1} \cdots D^{i_{N+M}}{}_{j_{N+M}} \, u^{j_1 \dots j_{N+M}}}} \qquad (*)$$

$\to$ <u>u is a rank-$(N+M)$ tensor</u>

Now suppose s is a pseudotensor and t a tensor, or vice versa. Then the rhs of $(*)$ gets multiplied

1 by $\det D$ $\to$ <u>u is a pseudotensor</u>

If both s and t are pseudotensors, then the rhs of $(*)$

1 gets multiplied by $(\det D)^2 = 1$ $\to$ <u>u is a tensor</u>

This proves prop. 1 from § 5.2.

Now consider

$$u^{k_1 \dots k_N} = t_i{}^{i k_1 \dots k_N} = g_{ij} \, t^{ij k_1 \dots k_N}$$

Let t be a tensor. Then

$$\underline{\tilde{u}^{k_1 \dots k_N}} = \tilde{g}_{ij} \, \tilde{t}^{ij k_1 \dots k_N}$$

1 Problem 2.5 $\downarrow$

$$= ((D^{-1})^T)_i{}^o ((D^{-1})^T)_j{}^p \, g_{op} \, D^i{}_m D^j{}_n D^{k_1}{}_{\ell_1} \cdots D^{k_N}{}_{\ell_N} \, t^{mn \ell_1 \dots \ell_N}$$

$$= \underbrace{(D^{-1})^o{}_i D^i{}_m}_{= \delta^o{}_m} \underbrace{(D^{-1})^p{}_j D^j{}_n}_{= \delta^p{}_n} D^{k_1}{}_{\ell_1} \cdots D^{k_N}{}_{\ell_N} \, g_{op} \, t^{mn \ell_1 \dots \ell_N}$$

$$= \Lambda^{k_1}{}_{l_1} \cdots \Lambda^{k_N}{}_{l_N}\, g_{mn}\, t^{mn\, l_1 \cdots l_N}$$

$$= \Lambda^{k_1}{}_{l_1} \cdots \Lambda^{k_N}{}_{l_N}\, u^{l_1 \cdots l_N} \qquad (**)$$

$\rightsquigarrow$ <u>u is a tensor of rank N</u>

If t is a pseudotensor, then the rhs of (**) gets multiplied by det Λ

$\rightsquigarrow$ <u>u is a pseudotensor of rank N</u>

(1) This proves prop 2 from [5]