

Problem Assignments # 8

11/15/2023
due 11/22/2023

2.3.1. Laurent series

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter the theorem in ch. 2 §3.2.

(3 points)

2.3.2. Applications of the residue theorem

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx \frac{1}{x^4 + 1}$$

b)

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

c)

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x} \frac{1}{1 + x^2}$$

and check your results by means of Wolfram Alpha.

Let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Use the residue theorem to show that

d)

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

e)

$$\int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{x^2 + a^2}$$

and define its Cauchy principal value by

$$\lim_{R \rightarrow 0} \left[\int_{-\infty}^{-R} dx f(x) + \int_R^{\infty} dx f(x) \right]$$

with $f(x) = 1/x(x^2 + a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \rightarrow 0$.

(17 points)
... /over

2.3.3. Matsubara frequency sum

Let $f(z)$ have simple poles at z_j ($j = 1, 2, \dots$), and no other singularities. Let $f(|z| \rightarrow \infty)$ go to zero faster than $1/z$. Consider the infinite sum

$$S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$$

with $\Omega_n = 2\pi Tn$ and $T > 0$. Show that

$$S = \sum_j n(z_j) \text{Res } f(z_j)$$

where $n(z) = 1/(e^{z/T} - 1)$ is the Bose distribution function.

hint: Show that $n(z)$ has simple poles at $z = i\Omega_n$, and integrate $n(z)f(z)$ over an infinite circle centered on the origin.

note: Sums of this form are important in finite-temperature quantum field theory. In this context, T is the temperature and Ω_n is called a “bosonic Matsubara frequency”.

(3 points)

2.3.1. Laurent series

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(3 points)

Solution:

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)} = \frac{1}{z - i} \frac{1}{2i + (z - i)} = \frac{1}{z - i} \frac{1}{2i} \frac{1}{1 + \frac{1}{2i}(z - i)}$$

1 pt

Now expand

$$\begin{aligned} \frac{1}{1 + \frac{1}{2i}(z - i)} &= \sum_{n=0}^{\infty} (-)^n \frac{(z - i)^n}{(2i)^n} \\ \Rightarrow f(z) &= \frac{1}{z - i} \frac{1}{2i} \sum_{n=0}^{\infty} (-)^n \frac{(z - i)^n}{(2i)^n} = \sum_{n=0}^{\infty} (-)^n \frac{(z - i)^{n-1}}{(2i)^{n+1}} = \sum_{n=-1}^{\infty} \frac{(-)^{n+1}}{(2i)^{n+2}} (z - i)^n \end{aligned}$$

1 pt

\Rightarrow The coefficients f_n in the Laurent series are

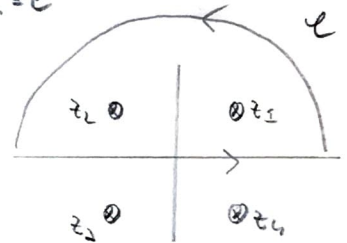
$$f_n = \begin{cases} (-)^{n+1}/(2i)^{n+2} & \text{for } n \geq -1 \\ 0 & \text{for } n < -1 \end{cases}$$

1 pt

2.2.2.) a)
$$I = \int_{-\infty}^{\infty} dx \frac{1}{x^4 + 1}$$

Wir schreiben $f(z) = \frac{1}{z^4 + 1} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

poles: $z^4 = -1 \rightarrow z_1 = e^{i\pi/4}, z_2 = e^{3i\pi/4}$
 $z_3 = e^{5i\pi/4}, z_4 = e^{7i\pi/4}$



$$\rightarrow I = \int_C dz f(z)$$

$$= 2\pi i [\text{Res } f(z_1) + \text{Res } f(z_2)]$$

$$\text{Res } f(z_1) = (z-z_1) f(z) \Big|_{z=z_1} = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{1}{(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})} = \frac{e^{-i\pi/4}}{(1-i)(1+i)(1+i)}$$

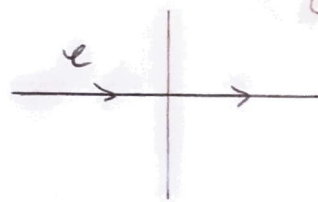
$$= \frac{1}{4} e^{-i\pi/4}$$

$$\text{Res } f(z_2) = \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{e^{-i3\pi/4}}{(1+i)(1-i)(1+i)} = \frac{1}{4} e^{-i3\pi/4}$$

$$\rightarrow I = 2\pi i \frac{1}{4} e^{-i\pi/4} (1 + e^{-i\pi/2}) = \frac{i\pi}{2} \frac{1}{\sqrt{2}} (1-i)^2$$

$$= \frac{i\pi}{2\sqrt{2}} (1-i)^2 = \frac{\pi}{\sqrt{2}}$$

b)
$$\underline{\underline{f}} = \int dx \frac{w(x)}{x} = \int dt \frac{w(t)}{t}$$



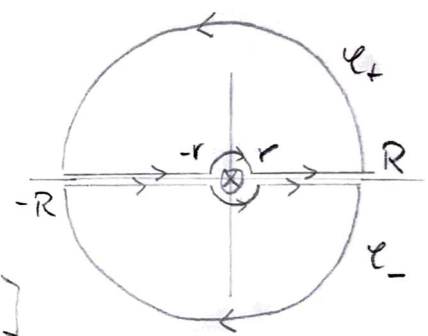
$$= \frac{1}{2i} \int_{\gamma} dt \frac{1}{t} [e^{it} - e^{-it}]$$

Remark: $\frac{w(t)}{t}$ is analytic everywhere

$\frac{1}{t} e^{\pm it}$ has a simple pole at $t=0$ with residue ± 1 .

Wieder

$$\underline{\underline{J}}_{\pm}(R,r) := \frac{1}{2i} \int_{\gamma_{\pm}} dt \frac{1}{t} e^{\pm it}$$



$$= \frac{1}{2i} \left[\int_{-R}^r dx \frac{1}{x} e^{\pm ix} + \int_r^R dx \frac{1}{x} e^{\pm ix} \right]$$

$$+ \frac{1}{2i} \int_{C_+} dt \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_-} dt \frac{1}{t} e^{\pm it}$$

$$=: J_{\pm}(R,r) + \frac{1}{2i} \int_{C_+} dt \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_-} dt \frac{1}{t} e^{\pm it}$$

Wann

$$\underline{\underline{J}}_{\pm}(R,r) = \int_{-R}^{-r} dx \frac{1}{x} e^{\pm ix} + \int_r^R dx \frac{1}{x} e^{\pm ix}$$

ad C_+ = Kreisbogen mit Radius r

C_- = Kreisbogen mit Radius R

$$\underline{\underline{f}} = \lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} [J_+(R,r) - J_-(R,r)]$$

Parametrisation

$$\left. \begin{aligned} C_+ : z = r e^{i\varphi} \quad (0 \leq \varphi \leq \sigma) \\ C_- : z = r e^{i\varphi} \quad (-\sigma \leq \varphi \leq 0) \end{aligned} \right\} dz = i r e^{i\varphi} d\varphi$$

$$\left. \begin{aligned} C'_+ : z = R e^{i\varphi} \quad (0 \leq \varphi \leq \sigma) \\ C'_- : z = R e^{i\varphi} \quad (0 \geq \varphi \geq -\sigma) \end{aligned} \right\} dz = i R e^{i\varphi} d\varphi$$

$$\Rightarrow \int_{C_+} dz \frac{1}{z} e^{iz} = i r \int_{-\sigma}^{\sigma} d\varphi e^{i\varphi} \frac{1}{r e^{i\varphi}} e^{i r e^{i\varphi}} = -i \int_{\sigma}^0 d\varphi e^{i r e^{i\varphi}}$$

$$\stackrel{r \rightarrow \infty}{=} -i\sigma + O(r)$$

$$\int_{C_-} dz \frac{1}{z} e^{-iz} = i r \int_{-\sigma}^0 d\varphi e^{i\varphi} \frac{1}{r e^{i\varphi}} e^{-i r e^{i\varphi}} = i\sigma + O(r)$$

$$\int_{C'_+} dz \frac{1}{z} e^{iz} = i R \int_0^{\sigma} d\varphi e^{i\varphi} \frac{1}{R e^{i\varphi}} e^{i R (\cos\varphi + i \sin\varphi)} = i \int_0^{\sigma} d\varphi e^{i R \cos\varphi - R \sin\varphi}$$

$$\int_{C'_-} dz \frac{1}{z} e^{-iz} = -i R \int_0^{\sigma} d\varphi e^{i\varphi} \frac{1}{R e^{i\varphi}} e^{i R \cos\varphi + R \sin\varphi} = i \int_0^{\sigma} d\varphi e^{i R \cos\varphi - R \sin\varphi}$$

$$\Rightarrow \left| \int_{C'_\pm} dz \frac{1}{z} e^{\pm iz} \right| = \left| \int_0^{\sigma} d\varphi e^{i R \cos\varphi - R \sin\varphi} \right| \leq \int_0^{\sigma} d\varphi \underbrace{|e^{i R \cos\varphi}|}_{=1} \cdot e^{-R \sin\varphi}$$

$$= 2 \int_0^{\sigma/2} d\varphi e^{-R \sin\varphi} = 2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} e^{-R x}$$

$$= \frac{2}{R} \int_0^R dx \frac{1}{\sqrt{1-x^2/R^2}} e^{-x} = \frac{2}{R} \left[\int_0^{\infty} dx e^{-x} + O(e^{-R}) \right]$$

$$= \frac{2}{R} [1 + O(e^{-R})] \rightarrow 0 \text{ for } R \rightarrow \infty$$

Now, $\tilde{f}_{\pm}(R, r) = 0$ by the residue theorem

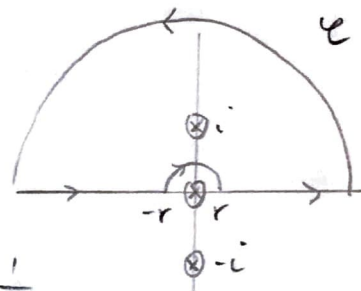
$$\rightarrow \underline{\underline{f}} = \lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} [\tilde{f}_+(R, r) - \tilde{f}_-(R, r)]$$

$$= \lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \left[\tilde{f}_+(R, r) - \frac{1}{2i} \int_{C_+} dt \frac{1}{t} e^{it} - \frac{1}{2i} \int_{C_+} dt \frac{1}{t} e^{it} \right. \\ \left. - \tilde{f}_-(R, r) + \frac{1}{2i} \int_{C_-} dt \frac{1}{t} e^{-it} + \frac{1}{2i} \int_{C_-} dt \frac{1}{t} e^{-it} \right]$$

$$= 0 - \frac{1}{2i} (-i\pi) - 0 - 0 + \frac{1}{2i} i\pi + 0 = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$

①

$$\begin{aligned}
 c) \quad \underline{I} &= \int_{-\infty}^{\infty} dx \frac{ix}{x} \frac{1}{x^2+1} = \int_{-\infty}^{\infty} dx \frac{1}{x} \frac{1}{zi} (e^{ix} - e^{-ix}) \frac{1}{x^2+1} \\
 &= -i \int_{-\infty}^{\infty} dx \underbrace{\frac{e^{ix}}{x} \frac{1}{x^2+1}} = -i \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x(x+i)(x-i)} \\
 &= -i \int_{-\infty}^{\infty} dx f(x)
 \end{aligned}$$



①

$f(|z| \rightarrow 0)$ für $\operatorname{Im} z \geq 0$

$$\begin{aligned}
 \rightarrow \int_{-\infty}^{\infty} dz f(z) &= 2\pi i \operatorname{Res} f(z=i) = 2\pi i \frac{e^{-1}}{i} \frac{1}{2i} \\
 &= -i\pi \frac{1}{e} \\
 \int_{-\infty}^{\infty} dx f(x) + \lim_{r \rightarrow 0} \int_{\text{small circle}} dz f(z) &= -i\pi \frac{1}{e}
 \end{aligned}$$

$$\rightarrow \int_{-\infty}^{\infty} dx f(x) = -\frac{i\pi}{e} - \lim_{r \rightarrow 0} \int_{\text{small circle}} dz f(z)$$

①

Along the semi-circle, $z = r(\cos\varphi + i\sin\varphi)$
 with $r = \text{const.}$



$$\rightarrow dz = r(-\sin\varphi + i\cos\varphi) d\varphi = ir(\cos\varphi + i\sin\varphi) d\varphi$$

$$\rightarrow \int_{\text{arc}} dz f(z) = \int_0^\pi d\varphi (\cos\varphi + i\sin\varphi) \frac{e^{ir(\cos\varphi + i\sin\varphi)}}{r(\cos\varphi + i\sin\varphi)} [1 + O(r)]$$

$$= -i\pi + O(r)$$

$$\rightarrow \int_{-\infty}^{\infty} dx f(x) = -\frac{i\pi}{e} + i\pi = \frac{i\pi}{e} (e-1)$$

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$$\rightarrow \underline{I} = \frac{i\pi}{e} (e-1)$$

Integral



≡ Examples ↗ Random

Assuming "Integral" refers to a computation | Use as a general topic or a character or referring to a mathematical definition or a word Instead

- function to Integrate:
- lower limit:
- upper limit:

Also Include: variable

Definite Integral:

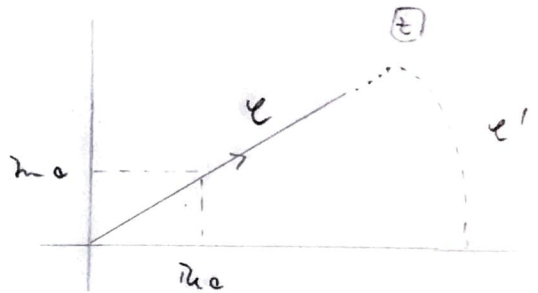
[More digits](#)

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2} dx = \frac{(e-1)\pi}{e} \approx 1.98587$$

$$d) \quad \Gamma := \int_{-\infty}^{\infty} dx e^{-ax^2} = 2 \int_0^{\infty} dx e^{-ax^2} \quad \text{Re } a > 0$$

$$t = \sqrt{a}x, \quad dt = \sqrt{a} dx$$

$$\rightarrow \Gamma = \frac{2}{\sqrt{a}} \int_0^{\infty} dt e^{-t^2}$$



On the other hand, the residue theorem yields

$$\int_{\gamma} dt e^{-t^2} + \int_{\gamma'} dt e^{-t^2} + \int_{\infty} dt e^{-t^2} = 0$$

$$\rightarrow \frac{1}{\sqrt{a}} \Gamma = \int_{\gamma} dt e^{-t^2} = - \int_{\gamma'} dt e^{-t^2} + \int_0^{\infty} dt e^{-t^2} = \int_0^{\infty} dx e^{-x^2}$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} dx e^{-x^2} = \frac{1}{\sqrt{a}} \left[\int_{-\infty}^{\infty} dx dx e^{-(x^2)^2} \right]^{1/2}$$

$$= \frac{1}{\sqrt{a}} \left(\int_0^{\infty} dx \int_0^{\infty} dx e^{-x^2} \right)^{1/2} = \frac{1}{\sqrt{a}} \left(\int_0^{\infty} dx e^{-x} \right)^{1/2} = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

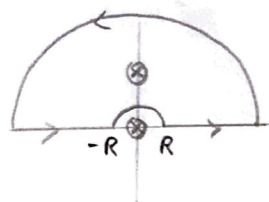
$$\rightarrow \Gamma = \frac{\sqrt{\pi}}{\sqrt{a}}$$

$$e) \quad \underline{f} = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{a^2+x^2} = \underline{\frac{1}{a^2} f}$$

$$f = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{1+x^2} = \int_{-\infty}^{\infty} \frac{dz}{z} \frac{1}{1+z^2}$$

remark: this integral exists only in the sense of a principal value

$$\underline{f} = \lim_{R \rightarrow 0} \int_C \frac{dz}{z} \frac{1}{1+z^2} - \lim_{R \rightarrow 0} \int_{\gamma} \frac{dz}{z} \frac{1}{1+z^2}$$



poles: $z=0, z=\pm i$

$$\text{Res} \left(\frac{1}{z} \frac{1}{1+z^2}, z=i \right) = \frac{1}{i} \frac{1}{2i} = -\frac{1}{2}$$

$$\begin{aligned} \Rightarrow \underline{f} &= 2\pi i \left(-\frac{1}{2} \right) - \lim_{R \rightarrow 0} \int_{\gamma} d\ell \, iR \frac{e^{i\ell}}{R e^{i\ell}} \frac{1}{1+R^2 e^{2i\ell}} = -\pi i - i \int_{\pi}^0 d\ell \\ &= -\pi i - i(-\pi) = \underline{0} \end{aligned}$$

$$\Rightarrow \underline{f} = \underline{0}$$

remark: This result is consistent with the usual principal value for an asymptotic integral:

$$\begin{aligned} \underline{\text{P.V.}}(f) &= \lim_{R \rightarrow 0} \left[\int_{-\infty}^{-R} \frac{dx}{x} \frac{1}{a^2+x^2} + \int_R^{\infty} \frac{dx}{x} \frac{1}{a^2+x^2} \right] = \\ &= \lim_{R \rightarrow 0} \int_R^{\infty} dx \left[\frac{1}{x} \frac{1}{a^2+x^2} - \frac{1}{x} \frac{1}{a^2+x^2} \right] = \underline{0} \end{aligned}$$

2.2.2.1) $\int_{-\infty}^{\infty} f(i\omega) d\omega$ with $\omega = 2\pi n$ ($n \in \mathbb{Z}$)

wieder $h(z) = \frac{1}{e^{z/T} - 1}$

$\rightarrow h(i\omega_n + \delta t) = \frac{1}{e^{i\omega_n T + \delta t/T} - 1} = \frac{1}{e^{\delta t/T} - 1}$

$= \frac{1}{\delta t/T + O((\delta t)^2)} = \frac{T}{\delta t} + O(1)$

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$\rightarrow h(z)$ has simple poles at $z = i\omega_n$ with residues T

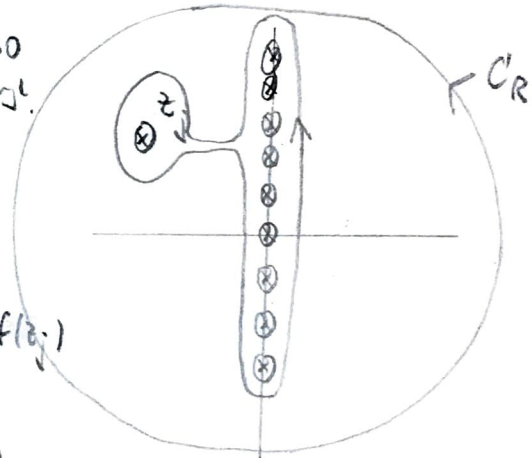
wieder

$f(z) \rightarrow 0$
fast enough

$\int = \lim_{R \rightarrow \infty} \oint_{C_R} \frac{dz}{2\pi i} h(z) f(z) = 0$

$= T \sum_n f(i\omega_n) + \sum_j h(z_j) \text{Res } f(z_j)$

$\rightarrow \int_{-\infty}^{\infty} f(\omega) d\omega = \sum_j h(z_j) \text{Res } f(z_j)$



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