PHYS 610 Mathematical Methods for Scientists F 2023

Problem Assignments # 9 11/22/2023
due 11/29/2023

2.4.1. **1-d Fourier transforms**

Consider a function f of one real variable x. Calculate the Fourier transforms $\hat{f}(k) = \int dx\, e^{-ikx}\, f(x)$ of the following functions:

a) $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

b) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

c) $f(x) = e^{-(x/x_0)^2}$.

(3 points)

2.4.2. **3-d Fourier transforms**

Consider a function f of one vector variable $\boldsymbol{x} \in I\!R^3$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x}) \quad .$$

Calculate the Fourier transforms of the following functions:

a)

$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{for } r < r_0 \\ 0 & \text{otherwise} \end{cases} \quad (r = |\boldsymbol{x}|) \quad .$$

b)

$$f(\boldsymbol{x}) = 1/r \quad .$$

hint: Consider $g(\boldsymbol{x}) = \frac{1}{r}\, e^{-r/r_0}$ and let $r_0 \to \infty$.

(3 points)

. . . **/over**

2.4.3. **More 1-d Fourier transforms**

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\ e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm i \int dt\, e^{izt}\, f_{\pm}(t) \qquad (\pm \text{ for } \mathrm{sgn}(\mathrm{Im}\, z) = \pm 1)$$

with z a complex frequency and $f_{\pm}(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\,[F(\omega + i0) - F(\omega - i0)] \qquad , \qquad F'(\omega) = \frac{1}{2}\,[F(\omega + i0) + F(\omega - i0)]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon \to 0} \epsilon/(x^2 + \epsilon^2) = \pi\, \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases $\int \frac{d\omega}{\pi}\, \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

2.4.1.) a) $\underline{\hat{f}(\lambda)} = \int dx\, e^{-i\lambda x}\,\Theta(|x|\le 1) = \int_{-1}^{1} dx\, e^{-i\lambda x} = \frac{-1}{i\lambda}\left(e^{-i\lambda x} - e^{i\lambda x}\right)$

$$= \frac{1}{i\lambda}\, 2i\sin\lambda = \underline{\underline{\frac{2}{\lambda}\sin\lambda}}$$

1

b) $\underline{\hat{f}(\lambda)} = \int_{-1}^{1} dx\, e^{-i\lambda x}(1-|x|) = \int_{-1}^{0} dx\, e^{-i\lambda x}(\cancel{1}+x) + \int_{0}^{1} dx\, e^{-i\lambda x}(\cancel{1}-x)$

$$= \frac{2}{\lambda}\sin\lambda + \frac{e^{-i\lambda x}}{-i\lambda}\left(x - \frac{1}{-i\lambda}\right)\Big|_{-1}^{0} - \frac{e^{-i\lambda x}}{-i\lambda}\left(x - \frac{1}{-i\lambda}\right)\Big|_{0}^{1}$$

$$= \cancel{\frac{2}{\lambda}\sin\lambda} - \cancel{\frac{1}{(i\lambda)^2}} + \frac{1}{i\lambda}e^{i\lambda}\left(-1+\frac{1}{i\lambda}\right) + \frac{1}{i\lambda}e^{-i\lambda}\left(1+\frac{1}{i\lambda}\right) - \cancel{\frac{1}{(i\lambda)^2}}$$

$$= \cancel{\frac{2}{\lambda}\sin\lambda} + \frac{2}{\lambda^2} - \cancel{\frac{1}{i\lambda}2i\sin\lambda} + \frac{1}{(i\lambda)^2}2\cos\lambda$$

$$= \underline{\underline{\frac{2}{\lambda^2}(1-\cos\lambda)}}$$

1

c) $\underline{\hat{f}(\lambda)} = \int dx\, e^{-i\lambda x}\, e^{-(x/x_0)^2} = x_0 \int dx\, e^{-x^2 - i\lambda x_0 x}$

$$= x_0 \int dx\, e^{-(x^2 + 2i(\lambda x_0/2)x - \lambda^2 x_0^2/4)}\, e^{-\lambda^2 x_0^2/4}$$

$$= x_0\, e^{-\lambda^2 x_0^2/4} \int dx\, e^{-(x + i\lambda x_0/2)^2}$$

$$= x_0\, e^{-\lambda^2 x_0^2/4} \int dx\, e^{-x^2}$$

1

$$= \underline{\underline{\sqrt{\pi}\, x_0\, e^{-\lambda^2 x_0^2/4}}}$$

2.4.2.) a) $f(\vec{x}) = f(r)$

$\rightsquigarrow$ w.l.g. choose $\vec{k} = (0,0,k)$

$$\rightsquigarrow \underline{\hat{f}(\vec{k})} = \hat{f}(k) = \int d\vec{x}\, e^{-ikz}\, \Theta(r_0 - r) = \int_0^{r_0} dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi\, e^{-ikr\cos\vartheta}$$

$$= 2\pi \int_0^{r_0} dr\, r^2 \frac{1}{-ikr}\left[e^{-ikr} - e^{ikr}\right]$$

$$= 2\pi \int_0^{r_0} dr\, r \frac{1}{\cancel{+i}k}\, 2\cancel{i} \sin kr = \frac{4\pi}{k} \int_0^{r_0} dr\, r \sin kr$$

$$= \frac{4\pi}{k^3} \int_0^{kr_0} dx\, x \sin x = \frac{4\pi}{k^3}\left[\sin x - x\cos x\right]_0^{kr_0}$$

(1) $$= \frac{4\pi}{k^3}\left[\sin kr_0 - kr_0 \cos kr_0\right]$$

b) $$\underline{\hat{g}(\vec{k})} = \hat{g}(k) = \int d\vec{x}\, e^{-ikr\cos\vartheta}\, \frac{1}{r} e^{-r/r_0} = 2\pi \int_0^{\infty} dr\, r^{\cancel{2}} \int_{-1}^{1} d\cos\vartheta\, e^{-ikr\cos\vartheta} \times \frac{1}{\cancel{r}} e^{-r/r_0}$$

$$= \cancel{2\pi} \int_0^{\infty} dr\, r\, e^{-r/r_0} \frac{1}{+\cancel{i}kr}\, \cancel{2i} \sin kr$$

$$= \frac{4\pi}{k} \int_0^{\infty} dr\, \sin kr\, e^{-r/r_0} = \frac{4\pi r_0}{k} \int_0^{\infty} dx\, e^{-x} \sin\left(x(kr_0)\right)$$

$$= \frac{4\pi r_0}{k} \frac{-1}{1+(kr_0)^2}\left[\sin kr_0 x + kr_0 \cos kr_0 x\right] e^{-x}\Big|_0^{\infty}$$

(1) $$= \frac{4\pi r_0}{\cancel{k}} \frac{1}{1+(kr_0)^2}\, \cancel{k} r_0 = \frac{4\pi r_0^2}{1+k^2 r_0^2}$$

$$\lim_{r_0\to\infty} g(\vec{x}) = f(\vec{x}) \rightsquigarrow \lim_{r_0\to\infty} \hat{g}(\vec{k}) = \hat{f}(\vec{k})$$

(1) $$\rightsquigarrow \hat{f}(\vec{k}) = \hat{f}(k) = \frac{4\pi}{k^2}$$

2.4.3.) In general, we have

$$F(\omega \pm i0) = \pm i \int dt\, \theta(\pm t)\, f(t)\, e^{i(\omega \pm i0)t} = \pm i \hat{f}_{\pm}(\omega)$$

with the $i0$ assuming convergence

a) $$\hat{f}_+(\omega) = \int_0^\infty dt\, e^{i\omega t} e^{-t/\tau} = \tau \int_0^\infty dt\, e^{-t(1-i\omega\tau)} = \frac{-\tau}{1-i\omega\tau}\Big|_0^\infty = \frac{\tau}{1-i\omega\tau}$$

1 $$\hat{f}_-(\omega) = \int_{-\infty}^0 dt\, e^{i\omega t} e^{t/\tau} = \int_0^\infty dt\, e^{-i\omega t} e^{-t/\tau} = \hat{f}_+(\omega)^* = \frac{\tau}{1+i\omega\tau}$$

$$\to F''(\omega) = \frac{-1}{2i}\left(i\hat{f}_+(\omega) + i\hat{f}_-(\omega)\right) = \frac{1}{2}\tau\left(\frac{1}{1-i\omega\tau} + \frac{1}{1+i\omega\tau}\right) = \frac{\tau}{1+\omega^2\tau^2}$$

$$F'(\omega) = \frac{1}{2}\left(i\hat{f}_+(\omega) - i\hat{f}_-(\omega)\right) = \frac{i}{2}\tau\left(\frac{1}{1-i\omega\tau} - \frac{1}{1+i\omega\tau}\right)$$

$$= \frac{i\tau}{\cancel{2}} \frac{\cancel{2}i\omega\tau}{1+\omega^2\tau^2} = \frac{-\omega\tau^2}{1+\omega^2\tau^2}$$

1 $$\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = 0 = F'(\omega=0)$$

b) $$F(\omega+i0) = i\int_0^\infty dt\, e^{i(\omega+i0)t} e^{i\omega_0 t} = i\int_0^\infty dt\, e^{i(\omega+\omega_0+i0)t} = \frac{-1}{\omega+\omega_0+i0}$$

1 $$F(\omega-i0) = \frac{-1}{\omega+\omega_0-i0}$$

$$\to F''(\omega) = \frac{-1}{2i}\left(\frac{1}{\omega+\omega_0+i0} - \frac{1}{\omega+\omega_0-i0}\right) = \frac{0}{(\omega+\omega_0)^2+0^2} = \pi\,\delta(\omega+\omega_0)$$

$$F'(\omega) = \frac{-1}{2}\left(\frac{1}{\omega+\omega_0+i0} + \frac{1}{\omega+\omega_0-i0}\right) = \frac{-1}{\omega+\omega_0}$$

1 $$\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = -\frac{1}{\omega_0} = F'(\omega=0)$$