

0.2.1 The brachistochrone problem

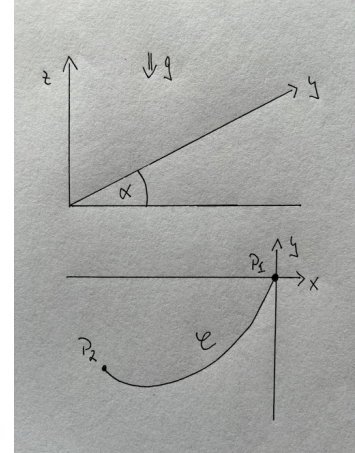
A point mass glides without friction on an inclined plane (inclination angle α) from point P_1 to point P_2 on a path \mathcal{C} according to Galilean mechanics.

- a) Use energy conservation to find the velocity as a function of y , using the coordinate system in the sketch.
- b) Write the passage time from P_1 to P_2 in the form

$$T = \int_{x_1}^{x_2} dx L(y, y')$$

with y considered a function of x and $y' = dy/dx$, and determine the Lagrangian L . Use the fact that Jacobi's integral is constant to find an ODE for y .

- c) Substitute $y' = \cot t$ to write the brachistochrone, i.e., the solution of the ODE from part b), in a parametric form, $y = y(t)$, $x = x(t)$.
- d) Express the passage time as a function of the value t_2 of the brachistochrone parameter in the point P_2 (or, equivalently, as a function of $y'_2 = (dy/dx)_{P_2}$, which has a more intuitive meaning).
- e) Find the passage time for the shortest path from P_1 to P_2 (as opposed to the brachistochrone) as a function of t_2 .
- f) Discuss the ratio of the two passage times as a function of t_2 .

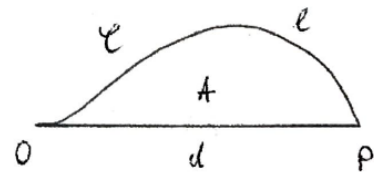


hint: The parameter value t_2 for the brachistochrone at the end point P_2 is a known function of y'_2 . It therefore suffices to discuss the passage time as a function of t_2 .

(18 points)

0.2.2 Dido's problem

An area A in the x - y -plane is enclosed by a straight line between two points O and P that are a distance d apart, and a path \mathcal{C} with end points O and P and length $\ell > d$. Find the path \mathcal{C} that maximizes A .



(6 points)

0.2.3 Geodesics on the 2-sphere

Show that the geodesics on the 2-sphere are great circles.

hint: There are various ways of doing this. One is to set up the problem of geodesics in \mathbb{R}_3 with the constraint that the desired paths $\vec{x}(t)$ must lie on the sphere. Now use the Euler-Lagrange equations for the constrained problem to show that $\vec{\ell} = \vec{x} \times \dot{\vec{p}} = \text{const}$, where $\dot{\vec{p}} = \partial L / \partial \dot{\vec{x}}$, with L the appropriate Lagrangian.

(5 points)