

0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta \varphi(x)} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(F[\varphi(y) + \epsilon \delta(y-x)] - F[\varphi(y)] \right)$$

a) Calculate $\delta F / \delta \varphi(x)$ for the following functionals:

i) $F = \int dx \varphi(x)$

ii) $F = \int dx \varphi^2(x)$

iii) $F = \int dx f(\varphi(x)) g(\varphi(x))$ where f and g are given functions

iv) $F = \int dx (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$

hint: Integrate by parts and assume that the boundary terms vanish.

v) $F = \int dx V(\varphi(x))$ where V is some given function.

remark: Blindly ignore terms that formally vanish as $\epsilon \rightarrow 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$ and an action $S = \int d^4x \mathcal{L}$. Show that extremizing S by requiring $\delta S / \delta \varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

(3 points)

0.2.5. **Massive scalar field**

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi(x)) (\partial^\mu \phi^*(x)) - \frac{m^2}{2} |\phi(x)|^2$$

with ϕ^* the complex conjugate of ϕ . What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \rightarrow \phi(x) e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(2 points)

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0.2.6. Particle in homogeneous \mathbf{E} and \mathbf{B} fields

Consider a point particle (mass m , charge e) in homogeneous fields $\mathbf{B} = (0, 0, B)$ and $\mathbf{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

- a) Show that the motion in z -direction decouples from the motion in the x - y plane, and find $z(t)$.
- b) Consider $\xi := x + iy$. Find the equation of motion for ξ , and its most general solution.

hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

- c) Show that the time-averaged velocity perpendicular to the plane defined by \mathbf{B} and \mathbf{E} is given by the *drift velocity*

$$\langle \mathbf{v} \rangle = c \mathbf{E} \times \mathbf{B} / B^2$$

Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

- d) Show that the path projected onto the x - y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.2.7. Harmonic oscillator coupled to a magnetic field

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0 , charge e). Put the oscillator in a homogeneous time-independent magnetic field $\mathbf{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to \mathbf{B} .

(4 points)