PHYS 622 Classical Electrodynamics W 2024

Problem Assignment # 3 01/31/2024
due 02/07/2024

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0.2.8. **Relativistic motion in parallel electric and magnetic fields**

Consider a relativistic charged particle (mass $m$, charge $e$) in parallel homogeneous electric and magnetic fields $\boldsymbol{E} = (0, 0, E)$, $\boldsymbol{B} = (0, 0, B)$.

a) Show that the equation of motion for the $z$-component of the momentum $p_z$ decouples from $p_x$ and $p_y$, and that the momentum perpendicular to the $z$-axis is a constant of motion: $p_x^2 + p_y^2 \equiv p_\perp^2 = \text{const.}$

b) Choose the zero of time such that $p_z(t = 0) = 0$, and show that with a suitable chosen origin the $z$-component of the particle's position can be written

$$z(t) = \frac{1}{eE} \sqrt{T_0^2 + c^2 e^2 E^2 t^2}$$

where $T_0$ is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time $t = 0$.

*hint:* Recall Einstein's law of falling bodies, ch. 0 §3.3.

c) Introduce a parameter $\varphi$ via $d\varphi/dt = ceB/T(t)$, with $T(t)$ the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$x = \frac{cp_\perp}{eB} \sin\varphi \quad , \quad y = \frac{cp_\perp}{eB} \cos\varphi \quad , \quad z = \frac{T_0}{eE} \cosh(E\varphi/B)$$

and explicitly find the relation between $\varphi$ and $t$.

*hint:* Consider $\pi := p_x + ip_y$ and note that $|\pi| = p_\perp = \text{const.}$ by the result of part a).

d) Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)

1.1.1. **Dual field tensor**

Show that the dual field tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$ obeys $\partial_\mu \tilde{F}^{\mu\nu}(x) = 0$.

*hint:* First show that $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$, and then relate $\partial_\mu \tilde{F}^{\mu\nu}(x)$ to that expression.

(2 points)

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1.1.2. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\mathrm{GL}} = \int d\boldsymbol{x} \Big[ r \left|\phi(\boldsymbol{x})\right|^2 + c \left| \left[\nabla - iq\boldsymbol{A}(\boldsymbol{x})\right] \phi(\boldsymbol{x}) \right|^2 + u \left|\phi(\boldsymbol{x})\right|^4 + \frac{1}{16\pi\mu} F_{ij}(\boldsymbol{x})\, F^{ij}(\boldsymbol{x}) \Big]$$

Here $\boldsymbol{x} \in \mathbb{R}^3$, and $\phi(\boldsymbol{x})$ is a complex-valued field that describes the superconducting matter, $\boldsymbol{A}$ is the Euclidian vector field that comprises the spatial components of the 4-vector $A^\mu = (A^0, \boldsymbol{A})$, and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2, 3)$. $\mu$ and $q$ are coupling constants that characterize the vector potential and its coupling to the matter, and $r$, $c$ and $u$ are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of $S_{\mathrm{GL}}$ with respect to all independent fields. (See Problem 0.2.4. You may want to double check against what you get from the Landau-Lifshitz method we used in class.)

b) Show that this theory is invariant under gauge transformations $\phi(x) \to \phi(\boldsymbol{x})\, e^{iq\lambda(\boldsymbol{x})}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\nabla}\lambda(\boldsymbol{x})$.

c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 0.2.5, can be made gauge invariant by coupling $\phi(x)$ to the electromagnetic vector potential $A^\mu(x)$.

*hint:* Replace the 4-gradient $\partial_\mu$ by $D_\mu = \partial_\mu - iqA_\mu$ and add the Maxwell Lagrangian.

*note:* If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)