

1.2.1. Energy-momentum tensor

Consider the electromagnetic field in the absence of matter.

- a) Show that the tensor field

$$H_{\mu}^{\nu}(x) = (\partial_{\mu} A_{\alpha}(x)) \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} A_{\alpha}(x))} - \delta_{\mu}^{\nu} \mathcal{L}$$

obeys the continuity equation

$$\partial_{\nu} H_{\mu}^{\nu}(x) = 0 \quad (*)$$

note: Notice that $H_{\mu}^{\nu}(x)$ is a generalization of Jacobi's integral in Classical Mechanics.

- b) Show that (*) also holds for

$$\tilde{T}_{\mu}^{\nu} = H_{\mu}^{\nu} + \partial_{\alpha} \psi_{\mu}^{\nu\alpha}$$

where $\psi_{\mu}^{\nu\alpha}$ is any tensor field that is antisymmetric in the second and third indices, $\psi_{\mu}^{\nu\alpha}(x) = -\psi_{\mu}^{\alpha\nu}(x)$.

- c) Show that $\psi_{\mu}^{\nu\alpha}$ can be chosen such that $\tilde{T}_{\mu}^{\nu}(x) = T_{\mu}^{\nu}(x)$, which provides an alternative proof that $T_{\mu}^{\nu}(x)$ obeys (*).

(5 points)

1.2.2. Energy-momentum conservation in the presence of matter

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

$$\partial_{\nu} T_{\mu}^{\nu}(x) = \frac{-1}{c} F_{\mu}^{\nu}(x) J_{\nu}(x)$$

(2 points)

1.2.3. Energy-momentum tensor for a massive scalar field

Consider the massive scalar field from ch. 0 §2.5:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \varphi) (\partial^{\mu} \varphi) - \frac{m^2}{2} \varphi^2$$

and the tensor field H_{μ}^{ν} defined analogously to Problem 1.2.1:

$$H_{\mu}^{\nu} = (\partial_{\mu} \varphi) \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \varphi)} - \delta_{\mu}^{\nu} \mathcal{L}$$

Determine H_{μ}^{ν} explicitly and show that

$$\partial_{\nu} H_{\mu}^{\nu} = 0$$

hint: Use the Euler-Lagrange equation determined in ch. 0 §2.5.

(2 points)

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1.2.4. Coulomb gauge

Consider the 4-vector potential $A^\mu(x) = (\varphi(x), \mathbf{A}(x))$. Show that one can always find a gauge transformation such that

$$\nabla \cdot \mathbf{A}(x) = 0$$

This choice is called *Coulomb gauge*.

(2 points)