Problem Assignment # 5 / Midterm 01/31/2024
due 02/21/2024

This assignment is in lieu of Assignment # 5. It doubles as the Midterm and overlaps with Assignments # 3 and 4, so you have three weeks to work on it. (I recommend that you solve Problem 0.2.8 before you start thinking about this one.) You can use any inanimate resource you like, but please don't consult any live resources. Before you start you may want to familiarize yourself again with the classical Kepler problem; see PHYS 611, or L&L Vol. I, or your favorite Classical Mechanics book.

0.2.9 Relativistic Kepler problem

Consider the motion of a point mass m in an attractive Coulomb potential centered at the origin within the framework of special relativity:

$$L = m\,c^2 - m\,c^2\sqrt{1 - (\vec{v}/c)^2} + \alpha/r \quad , \quad (\alpha > 0\ ,\ r = |\vec{x}|) \quad .$$

Let the z-component of the angular momentum be $\ell \geq 0$.

a) Use the conservation laws for the energy and the angular momentum to find the radial equation of motion,

$$\dot{r}^2 = c^2 f(r; E, \ell)$$

and the equation for the sectorial velocity

$$r^2\,\dot{\phi} = g(r; E, \ell)$$

where ϕ is the azimuthal angle. Determine the functions f and g explicitly and show that in the nonrelativistic limit they correctly reduce to the Galilean case.

hint: Take the angular momentum $\vec{\ell}$ to point in the z-direction. You know from Mechanics that the rotational invariance of L implies that the orbit lies in a plane perpendicular to $\vec{\ell}$; you can use that without proof. Use polar coordinates in the orbital plane.

b) Assume $\ell = 0$, and let $\dot{r}(t = 0) = 0$, $r(t = 0) = 2a$. Discuss and draw $\dot{r}$ as a function of r, and compare with the Galilean case. Determine the oscillation period $T(a)$ in terms of a dimensionless integral that depends only on the parameter $\xi = \alpha/2amc^2$. Discuss your result in the nonrelativistic and ultrarelativistic limits ($\xi \to 0$ and $\xi \to \infty$, respectively). Show that Kepler's third law gets modified within special relativity, and plot $T/T_{\xi=0}$ as a function of ξ.

c) Now assume $\ell > 0$. Use $f(r; E, \ell)$ from part a) to discuss all possible types of motion. Show that there are qualitative changes compared to Galilean mechanics; in particular that the particle can reach the center provided $\zeta := \alpha^2/\ell^2 c^2 > 1$. Determine the pericenter (distance closest to the center) and the apocenter (distance farthest from the center) and show that in the Keplerian result is recovered in the nonrelativistic limit $\zeta \to 0$. Find the condition for the existence of an allowed region and again show that the Keplerian result is recovered for $\zeta \to 0$.

hint: Write f in the form

$$f(r; E, \ell) = \frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2} - \zeta\,\frac{(p/r)^2}{(1 + \epsilon)^2}$$

with $p = \ell^2/m\alpha$ the quantity known as 'parameter' in celestial mechanics and $\epsilon = (E + \alpha/r)/mc^2$ a local energy parameter. Discuss ϵ as a function of r and the two contributions to f as functions of ϵ to obtain a qualitative picture of f as a function of r. Then solve the condition $f(r_\pm; E, \ell) = 0$ to obtain the turning points of the radial motion.

. . . **/over**

d) Assume $\zeta < 1$ and determine the equation of orbit. Show that the result can be written in a closed form that generalizes the Galilean result:

$$p^*/r = 1 + e^* \cos(\sqrt{1-\zeta}\,\phi) \quad .$$

Determine p^* and e^* and convince yourself that in the nonrelativistic limit you recover the Keplerian orbit.

e) Use the preceding results to calculate the perihelion advance of Mercury to leading order in $1/c^2$. Estimate the accuracy of that approximation. Compare your result with the experimental result of 43"/century.

hint: The semi-major axis, period, and eccentricity, respectively, of Mercury's Galilean orbit are: $a = 5.79 \times 10^{12}$ cm, $T = 88$ days, and $e = 0.206$

f) Consider the classical motion of an electron in the field of a Th nucleus ($\alpha = Ze^2$, $Z = 90$). Draw the orbit on a scale of $1 : 10^{-10}$ for the four cases $\ell = \hbar, 2\hbar$, $E = \pm 12\,\mathrm{keV}$. (Use $\hbar = 10^{-27}$ erg, $\hbar c/e^2 = 137$.) Also draw the corresponding Galilean orbits.

(40 points)