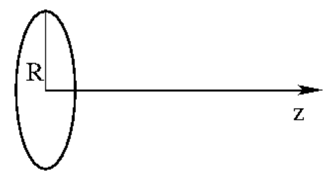
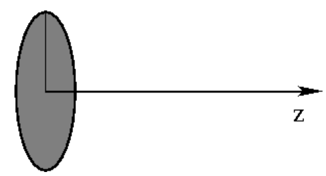
PHYS 622 Classical Electrodynamics W 2024

Problem Assignment # 7 02/28/2024
due 03/06/2024

2.2.1. **Planar charge distributions**



a) Consider a homogeneously charged infinitesimally thin ring with radius R and total charge Q that is oriented perpendicular to the z-axis. Calculate the electric field on the z-axis.



b) The same for a homogeneously charged disk with charge density σ and radius R. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

2.2.2. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\boldsymbol{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

hint: Make an *ansatz* for a purely radial field, $\boldsymbol{E}(\boldsymbol{x}) = E(r)\,\hat{e}_r$, and integrate Gauss's law over a spherical volume.

Explicitly calculate and plot the field $\boldsymbol{E}(\boldsymbol{x})$ and the potential $\varphi(\boldsymbol{x})$ for

b) a homogeneously charged sphere

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if} \quad r \leq r_0 \\ 0 & \text{if} \quad r > r_0 \ . \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(\boldsymbol{x}) = \sigma_0\, \delta(r - r_0) \ .$$

(8 points)

2.2.4. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \boldsymbol{\nabla}^2)\varphi(\boldsymbol{x}) = 4\pi\rho(\boldsymbol{x})$$

in terms of an integral.
hint: The answer is a generalization of Poisson's formula.

(3 points)

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2.3.1. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r} Q + \frac{1}{r^3} \boldsymbol{x} \cdot \boldsymbol{d} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \ldots$$

with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from part c) and calculate the quadrupole moments Q_{2m} as defined in ch.2 §3.5.

(10 points)