Problem Assignment # 2

01/19/2023
due 01/26/2023

0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F[\varphi(y)+\epsilon\delta(y-x)] - F[\varphi(y)]\right)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

i) $F = \int dx\, \varphi(x)$

ii) $F = \int dx\, \varphi^2(x)$

iii) $F = \int dx\, f(\varphi(x))\, g(\varphi(x))$ where f and g are given functions

iv) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
hint: Integrate by parts and assume that the boundary terms vanish.

v) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

0.2.5. **Massive scalar field**

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi(x))\,(\partial^\mu\phi^*(x)) - \frac{m^2}{2}\,|\phi(x)|^2$$

with ϕ^* the complex conjugate of ϕ. What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(2 points)

0.2.6. **Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields**

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

   hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\, \boldsymbol{E} \times \boldsymbol{B} / \boldsymbol{B}^2$$

   Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

0.2.7. **Harmonic oscillator coupled to a magnetic field**

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0, charge e). Put the oscillator in a homogeneous time-independent magnetic field $\boldsymbol{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $\boldsymbol{B}$.

(4 points)

0.2.4. **Functional derivative**

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$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)]\right)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

   i) $F = \int dx\, \varphi(x)$

   ii) $F = \int dx\, \varphi^2(x)$

   iii) $F = \int dx\, f(\varphi(x))\, g(\varphi(x))$ where f and g are given functions

   iv) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
   hint: Integrate by parts and assume that the boundary terms vanish.

   v) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

   remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

Solution

a) i) $\frac{\delta F}{\delta\varphi(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy\, [\varphi(y) + \epsilon\delta(y-x) - \varphi(y)] = \int dy\, \delta(y-x) = 1$

   ii) $\frac{\delta F}{\delta\varphi(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy \left[(\varphi(y) + \epsilon\delta(y-x))^2 - \varphi(y)^2\right] = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy\, \left[2\epsilon\varphi(y)\delta(y-x) + O(\epsilon^2)\right]$
   $= 2\varphi(x)$

   iii) $\frac{\delta F}{\delta\varphi(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy\, [f\,(\varphi(y) + \epsilon\delta(y-x))]\, [g\,(\varphi(y) + \epsilon\delta(y-x))]$
   $= f'(\varphi(x))g(\varphi(x)) + f(\varphi(x))g'(\varphi(x))$ 1pt

   iv) $\frac{\delta F}{\delta\varphi(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy \left[\left(\varphi'(y) + \epsilon\frac{d}{dy}\,\delta(y-x)\right)^2 - (\varphi'(y))^2\right] = 2\int dy\, \varphi'(y)\frac{d}{dy}\,\delta(y-x) = -2\varphi''(x)$

   v) $\frac{\delta F}{\delta\varphi(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy \left[V\left(\varphi'(y) + \epsilon\frac{d}{dy}\,\delta(y-x)\right) - V(\varphi'(y))\right]$
   $= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dy \left[\epsilon V'(\varphi'(y))\frac{d}{dy}\,\delta(y-x) + O(\epsilon^2)\right] = -V''(\varphi'(x))\,\varphi''(x)$ 1pt

b) $0 = \frac{\delta}{\delta\varphi(x)} \int d^4y\, \mathcal{L}\left(\varphi(y), \partial_\mu\varphi(y)\right)$
   $= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int d^4y\, \left[\mathcal{L}\left(\varphi(y) + \epsilon\delta(y-x), \partial_\mu\varphi(y) + \epsilon\partial_\mu\delta(y-x)\right) - \mathcal{L}\left(\varphi(y), \partial_\mu\varphi(y)\right)\right]$
   $= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int d^4y\, \left[\epsilon\delta(y-x)\, \frac{\partial\mathcal{L}}{\partial\varphi(y)} + \epsilon\left(\partial_\mu\delta(y-x)\right)\, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(y))} + O(\epsilon^2)\right]$
   $= \frac{\partial\mathcal{L}}{\partial\varphi(x)} - \partial_\mu\, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))}$ 1pt

p-Q.2.5

3.) a) $\frac{\partial \mathcal{L}}{\ }$ =

0.2.5.) a) Treat $\phi(x)$ and $\phi^*(x)$ as independent fields.

Minimizing with respect to ϕ^* yields

$$\underline{\underline{(\partial_\mu \partial^\mu + m^2)\phi(x) = 0}}$$

and minimizing with respect to ϕ just yields the c.c.:

$$\underline{\underline{(\partial_\mu \partial^\mu + m^2)\phi^*(x) = 0}}$$

1

b) Under $\phi(x) \to \phi(x) e^{i\Lambda(x)}$ we have

$$|\phi(x)|^2 \to |\phi(x)|^2$$

and

$$\partial_\mu \phi(x) \to (\partial_\mu \phi(x)) e^{i\Lambda(x)} + i(\partial_\mu \Lambda(x))\phi(x) e^{i\Lambda(x)}$$

$$\partial^\mu \phi^*(x) \to (\partial^\mu \phi^*(x)) e^{-i\Lambda(x)} - i(\partial^\mu \Lambda(x))\phi^*(x) e^{-i\Lambda(x)}$$

$$\to \underline{\partial_\mu \phi(x) \partial^\mu \phi^*(x)} \to \partial_\mu \phi(x) \partial^\mu \phi^*(x) - i(\partial^\mu \Lambda(x))(\partial_\mu \phi(x))\phi^*(x)$$
$$+ i(\partial_\mu \Lambda(x))(\partial^\mu \phi^*(x))\phi(x)$$
$$+ (\partial_\mu \Lambda(x))(\partial^\mu \Lambda(x))|\phi(x)|^2$$

$$\underline{\neq \partial_\mu \phi \partial^\mu \phi^*}$$

1

$\to$ <u>The Lagrangian is not invariant</u>

0.2.6.) a) Eq. of motion: $m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

Let $\vec{B} = (0,0,B)$ and $\vec{E} = (0,E_y,E_z)$

$$\rightarrow \boxed{\begin{array}{l} m\ddot{x} = \frac{e}{c}\dot{y}B \\ \hline m\ddot{y} = eE_y - \frac{e}{c}\dot{x}B \\ \hline m\ddot{z} = eE_z \end{array}} \quad \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

1

(3) $\rightarrow$ $z(t) = z_0 + v_z^0 t + \frac{eE_z}{2m}t^2$

b) Define $\xi := x + iy$

(1) + i·(2) $\rightarrow$ $m\ddot{\xi} = ieE_y - i\frac{eB}{c}\dot{\xi}$

Define $\omega := \frac{eB}{mc}$ cyclotron frequency

$\rightarrow$ $\boxed{\ddot{\xi} + i\omega\dot{\xi} = i\frac{e}{m}E_y}$ (*)

Special solution of inhomogeneous eq: $\dot{\xi} = \frac{eE_y}{m\omega}$

General solution of homogeneous eq: $\dot{\xi} = a e^{-i\omega t}$ $(a \in \mathbb{C})$

1 $\rightarrow$ $\dot{\xi}(t) = a e^{-i\omega t} + eE_y/m\omega$ is the most general solution of (*).

c) With $a = be^{i\alpha}$, $b,\alpha \in \mathbb{R}$

$\rightarrow$ $\dot{\xi} = be^{-i(\omega-\alpha)t} + eE_y/(m\omega)$

$\rightarrow$ α just shifts the zero of time $\rightarrow$ $\alpha = 0$ w.l.o.g.

$\rightarrow \quad \dot{x} + i\dot{y} = b\cos\omega t - ib\sin\omega t + eE_y/m\omega$

$$\rightarrow \quad \boxed{\begin{array}{l} \dot{x} = b\cos\omega t + eE_y/m\omega \\ \dot{y} = -b\sin\omega t \end{array}} \qquad (**)$$

1

$\rightarrow \quad \langle \dot{y} \rangle = 0 \,, \quad \langle \dot{x} \rangle = eE_y/m\omega = cE_y/B$ time-averaged velocity

$$= \frac{cE_y B}{B^2} = c(\vec{E}\times\vec{B})_x/B^2$$

in general:

$$\langle \vec{v} \rangle = \frac{c}{B^2}\vec{E}\times\vec{B} \qquad \text{drift velocity}$$

condition for $v \ll c$: $\quad E_y/B \ll 1 \quad$ necessary and sufficient condition for non-relativistic approximation

1

d) Choose $x(t=0) = 0 = y(t=0)$ w.l.o.g.

$$(**) \rightarrow \quad \boxed{\begin{array}{l} x(t) = \frac{b}{\omega}\sin\omega t + \frac{cE_y}{B}t \\ y(t) = \frac{b}{\omega}(\cos\omega t - 1) \end{array}}$$

$\rightarrow$ The path is a trochoid.

To visualize it, put $\omega = 1$ and define $C = cE_y/B$.

$$\rightarrow \quad \boxed{\begin{array}{l} x(t) = b\sin t + Ct \\ y(t) = b(\cos t - 1) \end{array}}$$

This is the projection of the path onto the x-y plane

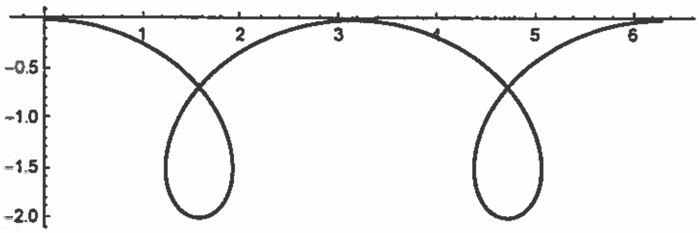
1

p0.2.6-7

For C < b the trochoid has loops :

In[21]:=
```
b = 1;
c = 0.5;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}]
```

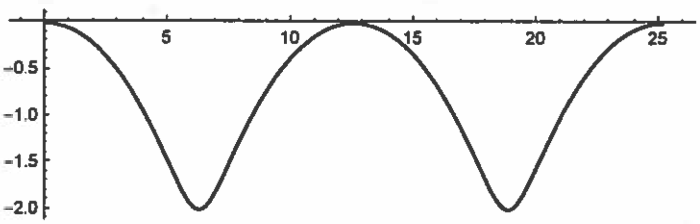
Out[25]=



For C > b it does not :

In[86]:=
```
b = 1;
c = 2;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

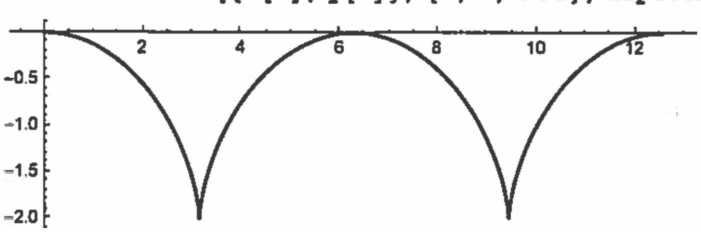
Out[90]=



And for C = b it degenerates into a cycloid :

In[91]:=
```
b = 1;
c = 1;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[95]=



1

0.2.7.) In addition to the restoring force $-m\omega_0^2 x$, the particle is subject to a Lorentz force

$$\frac{e}{c}\vec{v}\times\vec{B} = \frac{e}{c}(\dot{y}B, -\dot{x}B, 0)$$

→ The eqs of motion are

$$\boxed{\begin{aligned} \ddot{x} + \omega_0^2 x &= \Omega\dot{y} \qquad (1)\\ \ddot{y} + \omega_0^2 y &= -\Omega\dot{x} \qquad (2)\\ \ddot{z} + \omega_0^2 z &= 0 \qquad (3) \end{aligned}}$$

(1) with $\Omega := eB/mc$ the cyclotron frequency.

(3) → For oscillations in the z-direction, the frequency

(1) $\underline{\underline{\omega = \omega_0}}$ is unchanged

Define $\xi := x + iy$ and consider

(1)+i·(2) → $\boxed{\ddot{\xi} + \omega_0^2\xi = -i\Omega\dot{\xi}}$

(1)

ansatz: $\xi(t) = \xi_0 e^{i\omega t}$

→ $-\omega^2 + \omega_0^2 = \omega\Omega$

or $\omega^2 + \Omega\omega - \omega_0^2 = 0$

→ $\omega = \frac{1}{2}\left(-\Omega \pm \sqrt{\Omega^2 + 4\omega_0^2}\right) = \pm\sqrt{\omega_0^2 + \Omega^2/4} - \Omega/2$

→ The motion in the x-y plane is oscillatory, and the eigenfrequencies are

(1) $$\underline{\underline{\omega_\pm = \sqrt{\omega_0^2 + \Omega^2/4} \pm \Omega/2}}$$