Problem Assignment # 4

02/07/2024
due 02/14/2024

1.2.1. **Energy-momentum tensor**

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

$$H_\mu{}^\nu(x) = (\partial_\mu A_\alpha(x))\,\frac{\partial \mathcal{L}}{\partial\,(\partial_\nu A_\alpha(x))} - \delta_\mu{}^\nu\,\mathcal{L}$$

obeys the continuity equation

$$\partial_\nu\, H_\mu{}^\nu(x) = 0 \quad (*)$$

note: Notice that $H_\mu{}^\nu(x)$ is a generalization of Jacobi's integral in Classical Mechanics.

b) Show that (*) also holds for

$$\tilde{T}_\mu{}^\nu = H_\mu{}^\nu + \partial_\alpha \psi_\mu{}^{\nu\alpha}$$

where $\psi_\mu{}^{\nu\alpha}$ is any tensor field that is antisymmetric in the second and third indices, $\psi_\mu{}^{\nu\alpha}(x) = -\psi_\mu{}^{\alpha\nu}(x)$.

c) Show that $\psi_\mu{}^{\nu\alpha}$ can be chosen such that $\tilde{T}_\mu{}^\nu(x) = T_\mu{}^\nu(x)$, which provides an alternative proof that $T_\mu{}^\nu(x)$ obeys (*).

(5 points)

1.2.2. **Energy-momentum conservation in the presence of matter**

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

$$\partial_\nu T_\mu{}^\nu(x) = \frac{-1}{c}\,F_\mu{}^\nu(x)\,J_\nu(x)$$

(2 points)

1.2.3. **Energy-momentum tensor for a massive scalar field**

Consider the massive scalar field from ch. 0 §2.5:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\varphi)\,(\partial^\mu\varphi) - \frac{m^2}{2}\,\varphi^2$$

and the tensor field $H_\mu{}^\nu$ defined analogously to Problem 1.2.1:

$$H_\mu{}^\nu = (\partial_\mu\varphi)\,\frac{\partial \mathcal{L}}{\partial\,(\partial_\nu\varphi)} - \delta_\mu{}^\nu\,\mathcal{L}$$

Determine $H_\mu{}^\nu$ explicitly and show that

$$\partial_\nu H_\mu{}^\nu = 0$$

hint: Use the Euler-Lagrange equation determined in ch. 0 §2.5.

(2 points)

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1.2.4. **Coulomb gauge**

Consider the 4-vector potential $A^\mu(x) = (\varphi(x), \boldsymbol{A}(x))$. Show that one can always find a gauge transformation such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(x) = 0$$

This choice is called *Coulomb gauge*.

(2 points)

1.2.1.) a) $\underline{\partial_\nu \delta_\mu{}^\nu \mathcal{L}} = \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial A_\alpha} \partial_\mu A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)} \partial_\mu \partial_\lambda A_\alpha$

1 $\overset{EL}{=} \left(\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)}\right) \partial_\mu A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)} \partial_\mu \partial_\lambda A_\alpha$

$\underline{= \partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\alpha)} \partial_\mu A_\alpha}$

1 $\to \underline{\underline{0}} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)} \partial_\mu A_\alpha - \delta_\mu{}^\nu \mathcal{L} \right) = \underline{\underline{\partial_\nu t_\mu{}^\nu}}$

b) $\underline{\partial_\nu \partial_\alpha \psi_\mu{}^{\nu\alpha}} = -\partial_\nu \partial_\alpha \psi_\mu{}^{\alpha\nu} = -\partial_\alpha \partial_\nu \psi_\mu{}^{\nu\alpha} = \underline{-\partial_\nu \partial_\alpha \psi_\mu{}^{\nu\alpha}}$

1 $\to \underline{\partial_\nu \partial_\alpha \psi_\mu{}^{\nu\alpha} = 0} \quad \to \quad \underline{\underline{\partial_\nu \tilde{T}_\mu{}^\nu = 0}}$

1 c) Wenn $\psi^{\mu\nu\alpha} = \frac{1}{4\pi} A^\mu F^{\nu\alpha} = \frac{-1}{4\pi} A^\mu F^{\alpha\nu} = -\psi^{\mu\alpha\nu}$ ✓

$\to \partial_\nu \tilde{T}_\mu{}^\nu = 0$, und

$\underline{\underline{\tilde{T}^{\mu\nu}}} = t^{\mu\nu} + \partial_\alpha \psi^{\mu\nu\alpha} = (\partial^\mu A_\alpha) \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\alpha)} - g^{\mu\nu} \mathcal{L} + \frac{1}{4\pi} \partial_\alpha A^\mu F^{\nu\alpha}$

$\overset{\S 1.3}{=} (\partial^\mu A_\alpha) \frac{1}{4\pi} F^{\alpha\nu} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda} + \frac{1}{4\pi} (\partial_\alpha A^\mu) F^{\nu\alpha} + \frac{1}{4\pi} A^\mu \underbrace{\partial_\alpha F^{\nu\alpha}}_{=0}$

$= \frac{-1}{4\pi} (\partial^\mu A_\alpha - \partial_\alpha A^\mu) F^{\nu\alpha} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda}$

$= \frac{-1}{4\pi} F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda}$

$= \frac{-1}{4\pi} F^{\mu\alpha} F^\nu{}_\alpha + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda} = \underline{\underline{T^{\mu\nu}}}$

1 $\to \underline{\underline{\partial_\nu T_\mu{}^\nu = 0}}$

1.22.) Generalizing the proof of the proposition in l 1 §2.3:

The only difference is that now the EL eq. reads

(1) $$\partial_\nu F^\nu{}_\alpha = \frac{4\pi}{c} j_\alpha$$

$$\rightsquigarrow \underline{\partial_\nu T_\mu{}^\nu} \overset{§2.3}{=} \frac{1}{4\pi}\left[-(\partial_\nu F_\mu{}^\alpha) F^\nu{}_\alpha - F_\mu{}^\alpha \partial_\nu F^\nu{}_\alpha + \frac{1}{4}\partial_\mu F_{\alpha\beta}F^{\alpha\beta}\right]$$

$$= -\frac{1}{c} F_\mu{}^\alpha j_\alpha + \frac{1}{4\pi}\underbrace{\left[-(\partial_\nu F_\mu{}^\alpha)F^\nu{}_\alpha + \frac{1}{2}(\partial_\mu F_{\alpha\beta})F^{\alpha\beta}\right]}_{=0 \text{ by } §2.3}$$

$$= \underline{\underline{-\frac{1}{c} F_\mu{}^\nu j_\nu}}$$

(1)

1.2.3.) $\underline{H_\mu{}^\nu} = (\partial_\mu \varphi) \dfrac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} - \delta_\mu{}^\nu \mathcal{L}$

$$= \underline{(\partial_\mu\varphi)(\partial^\nu\varphi) - \delta_\mu{}^\nu \tfrac{1}{2}(\partial_\alpha\varphi)(\partial^\alpha\varphi) + \delta_\mu{}^\nu \tfrac{m^2}{2}\varphi^2}$$

1

$$\rightarrow \underline{\underline{\partial_\nu H_\mu{}^\nu}} = (\partial_\nu\partial_\mu\varphi)(\partial^\nu\varphi) + (\partial_\mu\varphi)(\partial_\nu\partial^\nu\varphi) - (\partial_\alpha\varphi)(\partial_\mu\partial^\alpha\varphi) + m^2\varphi\partial_\mu\varphi$$

$$= \underbrace{(\partial_\nu\partial_\mu\varphi)(\partial^\nu\varphi) - (\partial_\mu\partial_\alpha\varphi)(\partial^\alpha\varphi)}_{=0} + (\partial_\mu\varphi)(\partial_\nu\partial^\nu\varphi + m^2\varphi)$$

$$= (\partial_\mu\varphi)\underbrace{(\partial_\nu\partial^\nu + m^2)\varphi}_{=0} = \underline{\underline{0}}$$

by the Klein-Gordon eq. (cf. Q §2.5)

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1.2.4.) Gauge transfo: $A^\mu \to A^\mu - \partial^\mu \chi$

$$\to \vec{A} \to \vec{A} - \vec{\nabla}\chi$$

$$\to \vec{\nabla}\cdot\vec{A} \to \vec{\nabla}\cdot\vec{A} - \vec{\nabla}^2\chi$$

1

Now choose χ as any solution of the Poisson eq.

$$\vec{\nabla}^2\chi(x) = \vec{\nabla}\cdot\vec{A}(x)$$

Then the transformed $\vec{A}$ has the property

$$\underline{\underline{\vec{\nabla}\cdot\vec{A}'(x)}} = \vec{\nabla}\cdot\vec{A}(x) - \vec{\nabla}^2\chi(x) = \underline{\underline{0}}$$

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