PHYS 622 Classical Electrodynamics W 2024

Problem Assignment # 6 02/14/2024
due 02/21/2024

1.3.1. **Energy density**

Show that the argument from ch. 1 §3.6 remark (3) for $u(\boldsymbol{x}, t)$ being the energy density of the electromagnetic field still holds if the field is coupled to N relativistic particles rather than one nonrelativistic one.

(3 points)

1.3.2. **Addition of velocities**

Consider a particle that has a velocity $\boldsymbol{v}$ in some inertial frame. Find the velocity of the particle in another inertial frame that moves with a velocity $\boldsymbol{V}$ with respect to the first one. Use the result to show that the velocity in the second frame is less than c, provided it was less than c in the first one.

(3 points)

1.3.3. **Galileo transformations of Maxwell's equations**

a) Show explicitly which of Maxwell's equations are or are not invariant under Galileo transformations. *hint:* Consider the transformations of all vectors (the 4-gradient, the fields, and the 4-current) to zeroth order in $1/c$, but keep the terms of $O(1/c)$ in Maxwell's equations. In other words, note that if you do a Lorentz transformation consistently to a given order in $1/c$, then of course all of Maxwell's equations are invariant.

b) Suppose you had never heard of Lorentz transformations, but were familiar with Galilean mechanics. What are the two logical conclusions you could draw from the result of part a)? (Obviously, one of them by now is of historical interest only.)

(4 points)

1.3.4. **Lorentz transformations of fields**

Consider static and homogeneous fields $\boldsymbol{E}$ and $\boldsymbol{B}$ that are not parallel to one another in some inertial frame.

a) Show that there exists an inertial frame in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel, and that the two frames are related by a Lorentz boost whose velocity is given by the solution of the equation

$$\frac{\boldsymbol{V}}{c}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) = (1 + \boldsymbol{V}^2/c^2)\,\boldsymbol{E} \times \boldsymbol{B}$$

b) Show explicitly that this equation has one and only one physical solution that obeys $|\boldsymbol{V}|/c < 1$, that there always is a physical solution, and that the result in the limit of almost parallel fields in the original reference frame is sensible.

c) Are there other inertial frames in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are parallel? If so, how many?

(7 points)

12.2.) Consider one relativistic particle. Then the kinetic energy is

$$E_{ki} = \vec{v}\cdot\frac{\partial L_0}{\partial \vec{v}} - L_0 = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

Now consider

$$\vec{v}\cdot\dot{\vec{p}} = \vec{v}\cdot\frac{d}{dt}\frac{\partial L_0}{\partial \vec{v}} = \vec{v}\cdot\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \frac{m\vec{v}\cdot\dot{\vec{v}}}{\sqrt{1-v^2/c^2}} + mv^2\frac{\vec{v}\cdot\dot{\vec{v}}/c^2}{(1-v^2/c^2)^{3/2}}$$

(1)
$$= \frac{m\vec{v}\cdot\dot{\vec{v}}}{(1-v^2/c^2)^{3/2}} = \frac{d}{dt}E_{ki}$$

But from the eq. of motion we have

$$\dot{\vec{p}} = \vec{F} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$$

$$\rightsquigarrow \frac{d}{dt}E_{ki} = \vec{v}\cdot\dot{\vec{p}} = e\vec{v}\cdot\vec{E} \qquad \text{since } \vec{v}\cdot(\vec{v}\times\vec{B}) = 0$$

This is the same expression as in ch 1 §1.6 remark (2), and Poynting's theorem again yields

$$\frac{d}{dt}(U + E_{ki}) = 0 \qquad \text{where } U = \int d\vec{x}\, u(\vec{x},t)$$

(1) $\rightsquigarrow$ U is the field energy

For N particles with charges e_i $(i=1,\dots,N)$, consider

$$\int d\vec{x}\, \vec{j}\cdot\vec{E} = \sum_{i=1}^{N} e_i\vec{v}_i\cdot\vec{E}$$

$$\rightsquigarrow \frac{d}{dt}E_{ki} = -\sum_{i=1}^{N} e_i\vec{v}_i\cdot\vec{E}$$

(1) $\rightsquigarrow$ same result as for $N=1$

1.7.2.) cf. §4.1 → Lorentz boost in x-direction

$$\hat{x} = \gamma x + \gamma V t \quad , \quad \hat{t} = \gamma t + \gamma \frac{V}{c^2} x \qquad \underline{\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}}$$

$$\to \quad d\hat{x} = \gamma(dx + V dt) \quad , \quad d\hat{t} = \gamma\left(dt + \frac{V}{c^2} dx\right)$$

$$d\hat{y} = dy$$

$$d\hat{z} = dz$$

$$\to \underline{\hat{v}_x} = \frac{d\hat{x}}{d\hat{t}} = \frac{d\hat{x}}{dt} \frac{1}{d\hat{t}/dt} = \frac{\gamma v_x + \gamma V}{\gamma + \gamma V v_x/c^2} = \underline{\frac{v_x + V}{1 + v_x V/c^2}} \qquad (1)$$

$$\underline{\underline{\hat{v}_y}} = \frac{d\hat{y}}{d\hat{t}} = \frac{v_y}{\gamma(1 + v_x V/c^2)} = \underline{\underline{\frac{v_y \sqrt{1 - V^2/c^2}}{1 + v_x V/c^2}}} \qquad (2)$$

① $$\underline{\hat{v}_z} = \underline{\frac{v_z \sqrt{1 - V^2/c^2}}{1 + v_x V/c^2}} \qquad (3)$$

let $\hat{\beta}_x = \hat{v}_x / c \quad , \quad \beta = V/c$

$$\to \underline{1 - \hat{\beta}_x^2} = 1 - \frac{(\beta_x + \beta)^2}{(1 + \beta_x \beta)^2} = \frac{1 + \cancel{2\beta_x\beta} + \beta_x^2\beta^2 - \beta_x^2 - \beta^2 - \cancel{2\beta_x\beta}}{(1 + \beta_x\beta)^2} =$$

$$= \frac{(1 - \beta_x^2)(1 - \beta^2)}{(1 + \beta_x\beta)^2} \underline{\geq 0} \quad \text{provided } \beta_x, \beta \leq 1$$

and $1 - \hat{\beta}_x^2 = 0$ iff ($\beta_x = 1$ or $\beta = 1$)

① $$\to \hat{\beta}_x^2 \leq 1 \quad \to \quad \underline{\underline{|\hat{v}_x| \leq c}} \quad \text{and } |\hat{v}_x| = c \text{ iff } (|v_x| = c \text{ or } |V| = c)$$

Now define $v_\perp^2 := v_y^2 + v_z^2 \quad \to \quad v_x^2 + v_\perp^2 \leq c^2$

$$\to \quad \underline{v_\perp^2 \leq c^2 - v_x^2} \qquad (*)$$

$$(2),(3) \to \underline{\hat{v}_\perp^2} = v_\perp^2 \frac{1 - \beta^2}{(1 + \beta_x\beta)^2} \overset{(*)}{\leq} c^2 (1 - \beta_x^2) \frac{1 - \beta^2}{(1 + \beta_x\beta)^2}$$

$$\to \underline{1 - \hat{\beta}_\perp^2} := 1 - \frac{\hat{v}_\perp^2}{c^2} \geq 1 - \frac{(1 - \beta_x^2)(1 - \beta^2)}{(1 + \beta_x\beta)^2} = \frac{\cancel{1} + 2\beta_x\beta + \cancel{\beta_x^2\beta^2} - \cancel{1} + \beta_x^2 + \beta^2 - \cancel{\beta_x^2\beta^2}}{(1 + \beta_x\beta)^2}$$

① $$= \underline{\frac{(\beta_x + \beta)^2}{(1 + \beta_x\beta)^2} > 0} \quad \to \quad \underline{\hat{\beta}_\perp^2 < 1} \quad \to \quad \underline{\underline{|\hat{v}_\perp| \leq c}}$$

1.2.2 (i) a) Consider a Lorentz boost along the x-axis (cf 2 § 4.1):

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1+O(V^2/c^2) & V/c+O(V^2/c^2) & 0 & 0 \\ V/c+O(V^2/c^2) & 1+O(V^2/c^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and transform the 4-position:

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct + O(1/c) \\ Vt + x + O(1/c^2) \\ y \\ z \end{pmatrix}$$

$\rightsquigarrow$ To zeroth order in $1/c$, $\underline{\underline{\tilde{t} = t, \ \hat{\vec{x}} = \vec{x} + \vec{V}t}}$ ✓

Now transform the derivatives:

$$\tilde{\partial}_\mu = \begin{pmatrix} \frac{1}{c}\partial_{\tilde{t}} \\ \hat{\vec{\nabla}} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\partial_t + \frac{V}{c}\partial_x + \dots \\ \partial_x + \dots \\ \partial_y \\ \partial_z \end{pmatrix} \rightsquigarrow \underline{\underline{\partial_{\tilde{t}} = \partial_t + \vec{V}\cdot\vec{\nabla}, \ \hat{\vec{\nabla}} = \vec{\nabla}}}$$

to $O(1/c^0)$

(1)

The fields have no time-like component

$\rightsquigarrow$ $\underline{\underline{\hat{\vec{E}} = \vec{E}, \ \hat{\vec{B}} = \vec{B}}}$ to $O(1/c^0)$

(this also follows explicitly from cf 2 § 4.2).

Finally, the 4-current:

$$\hat{j}^\mu = \begin{pmatrix} c\tilde{\rho} \\ \hat{\vec{j}} \end{pmatrix} = \begin{pmatrix} c\rho + \dots \\ V\rho + \vec{j} + \dots \end{pmatrix} \rightsquigarrow \underline{\underline{\hat{\rho} = \rho, \ \hat{\vec{j}} = \vec{j} + \vec{V}\rho}}$$

to $O(1/c^0)$

(1)

Now consider Maxwell's eqs.:

(1) $\tilde{\vec{\nabla}}\cdot\tilde{\vec{B}} = \vec{\nabla}\cdot\vec{B} = 0$ ✓

(2) $\frac{1}{c}\partial_{\tilde{t}}\tilde{\vec{B}} + \tilde{\vec{\nabla}}\times\tilde{\vec{E}} = \underbrace{\frac{1}{c}\partial_t\vec{B} + \vec{\nabla}\times\vec{E}}_{=0} + \frac{1}{c}(\vec{V}\cdot\vec{\nabla})\vec{B} = \frac{1}{c}(\vec{V}\cdot\vec{\nabla})\vec{B} \neq 0$

(3) $\tilde{\vec{\nabla}}\cdot\tilde{\vec{E}} = \vec{\nabla}\cdot\vec{E} = 4\pi\rho = 4\pi\tilde{\rho}$ ✓

(4)
$$\begin{aligned} -\frac{1}{c}\partial_{\tilde{t}}\tilde{\vec{E}} + \tilde{\vec{\nabla}}\times\tilde{\vec{B}} &= -\frac{1}{c}\partial_t\vec{E} + \vec{\nabla}\times\vec{B} - \frac{1}{c}(\vec{V}\cdot\vec{\nabla})\vec{E} \\ &= \frac{4\pi}{c}\vec{j} - \frac{1}{c}(\vec{V}\cdot\vec{\nabla})\vec{E} \\ &= \frac{4\pi}{c}\tilde{\vec{j}} - \frac{4\pi}{c}\rho\vec{V} - \frac{1}{c}(\vec{V}\cdot\vec{\nabla})\vec{E} \neq \frac{4\pi}{c}\tilde{\vec{j}} \end{aligned}$$

→ (1) and (3) are covariant under Galileo boosts, but (2) and (4) are not.

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remark: (1) The eqs that contain a time derivative are not covariant. The "streaming term" $\vec{V}\cdot\vec{\nabla}$ in $\partial_{\tilde{t}}$ screws things up (and also the $\rho\vec{V}$ term in $\tilde{\vec{j}}$, which is a velocity term and hence equivalent to a time derivative.

(2) Another way to say it: M's eqs contain terms of $O(1/c)$ in (2) and (4), whereas the Galileo boost contains the transformation only to $O(1)$, so things are not consistent.

b) Two possibilities:

(1) Maxwell's eqs are valid only in a special reference frame known as "ether". Michelson-Morley killed that possibility.

(2) Newtonian mechanics is wrong. This turned out to be the resolution; special relativity fixed the problem.

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1.3.4) a) Let the non-parallel vectors $\vec{E}, \vec{B}$ lie in the y-z-plane:

$$\vec{E} = (0, E_y, E_z), \quad \vec{B} = (0, B_y, B_z)$$

Now consider a Lorentz boost in the x-direction

cf 2 §4.2 ~> $\hat{E}_x = E_x = 0$ and $\hat{B}_x = B_x = 0$.

$$\rightsquigarrow \hat{\vec{E}} = (0, \hat{E}_y, \hat{E}_z), \quad \hat{\vec{B}} = (0, \hat{B}_y, \hat{B}_z)$$

$$\rightsquigarrow \hat{\vec{E}} \times \hat{\vec{B}} = (\hat{E}_y \hat{B}_z - \hat{E}_z \hat{B}_y, 0, 0)$$

$$\rightsquigarrow \hat{\vec{E}} \parallel \hat{\vec{B}} \quad \text{iff} \quad \hat{E}_y \hat{B}_z - \hat{E}_z \hat{B}_y = 0$$

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cf 2 §4.2 ~>

$$\underline{0} = (E_y \cosh\phi + B_z \sinh\phi)(B_z \cosh\phi + E_y \sinh\phi) - (E_z \cosh\phi - B_y \sinh\phi)(B_y \cosh\phi - E_z \sinh\phi)$$

$$= (E_y B_z - E_z B_y)(\cosh^2\phi + \sinh^2\phi) + (E_y^2 + E_z^2 + B_y^2 + B_z^2)\cosh\phi \sinh\phi$$

$$\overset{\S 4.1}{=} \underline{\gamma^2 (1+\beta^2)(\vec{E}\times\vec{B})_x + \gamma^2 \beta (\vec{E}^2 + \vec{B}^2)} \qquad \text{where } \beta = V/c$$

If we make the orientation of the plane defined by $\vec{E}$ and $\vec{B}$ arbitrary, then $\vec{V}$ is given by the solution of

$$\underline{\underline{\frac{\vec{V}}{c}(\vec{E}^2 + \vec{B}^2) = (1 + \vec{V}^2/c^2)\, \vec{E}\times\vec{B}}}$$

1

b) Now return to the original geometry and denote

$$(\vec{E}\times\vec{B})_x =: x, \quad \vec{E}^2 + \vec{B}^2 =: s, \quad V_x =: V$$

$$\rightsquigarrow \quad x\beta^2 - s\beta + x = 0$$

$\rightsquigarrow \quad \Lambda = \frac{1}{2x}\left(s \pm \sqrt{s^2-4x^2} \right) \qquad (*)$

Now consider $\quad s^2-4x^2 = (E^2+B^2)^2 - 4E^2B^2\sin^2\alpha \qquad \alpha = \sphericalangle(\vec{E},\vec{B})$

$$\geq (E^2+B^2)^2 - 4E^2B^2$$

$$= (E^2-B^2)^2 \geq 0 \qquad (+)$$

$\rightsquigarrow \quad \sqrt{s^2-4x^2} \in \mathbb{R}$ always

1 $\rightsquigarrow$ There always is at least one solution

That (+) $\rightsquigarrow \left(\frac{s}{2x}\right)^2 \geq 1$

Let $x > 0$ w.l.o.g. $\rightsquigarrow \quad s/2x > 1$

$$\rightsquigarrow \quad \frac{s}{2x} + \frac{1}{2x}\sqrt{s^2-4x^2} > 1$$

$\rightsquigarrow$ The only candidate for a physical solution is

1 $$\Lambda = \frac{1}{2x}\left(s-\sqrt{s^2-4x^2}\right) = \frac{s}{2x} - \sqrt{\left(\frac{s}{2x}\right)^2-1}$$

We still need to show that this solution obeys $\Lambda < 1$.

Define $\quad \alpha := s/2x > 0$ w.l.o.g.

Then we know $\alpha > 1$.

Now demand $\quad \alpha - \sqrt{\alpha^2-1} \overset{!}{<} 1$

$\Leftrightarrow \quad \alpha - 1 < \sqrt{\alpha^2-1} \quad \Leftrightarrow \quad (\alpha-1)^2 < \alpha^2-1 \quad \Leftrightarrow \quad -2\alpha+1 < -1$

$\Leftrightarrow \quad \alpha > 1 \quad$ ✓

1 $\rightsquigarrow \quad \Lambda = \alpha - \sqrt{\alpha^2-1} \quad$ where $\quad \alpha = \frac{s}{2x} = \frac{E^2+B^2}{2|\vec{E}\times\vec{B}|} \quad$ is the unique physical solution which always exists

Consider almost parallel fields $\rightarrow$ $\vec{E}\times\vec{B} \rightarrow 0$ $\rightarrow$ $\alpha \rightarrow \infty$

$$\rightsquigarrow \underline{\underline{\Lambda}} \rightarrow \alpha - \alpha\sqrt{1-1/\alpha^2} = \alpha - \alpha + \frac{1}{2\alpha} + O(1/\alpha^3) = \frac{1}{2\alpha} + O(1/\alpha^3) \rightarrow \underline{\underline{0}}$$

which is sensible: If the fields are almost parallel to start with, then a small boost velocity suffices to make them parallel.

(1)

c) Once $\vec{E} \parallel \vec{B}$, e.g., $\vec{E} = (E_x,0,0)$, $\vec{B} = (B_x,0,0)$ w.l.o.g., then any Lorentz boost in the common direction of $\vec{E}$ and $\vec{B}$ will leave $\vec{E} \parallel \vec{B}$ $\rightsquigarrow$ <u>There are infinitely many such inertial frames</u>

(1)