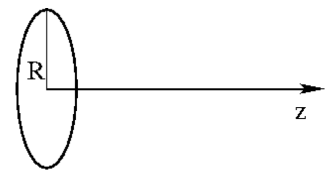
PHYS 622 Classical Electrodynamics W 2024

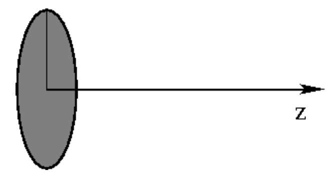
Problem Assignment # 7

02/28/2024
due 03/06/2024

2.2.1. **Planar charge distributions**



a) Consider a homogeneously charged infinitesimally thin ring with radius R and total charge Q that is oriented perpendicular to the z-axis. Calculate the electric field on the z-axis.



b) The same for a homogeneously charged disk with charge density σ and radius R. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

2.2.2. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\boldsymbol{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

hint: Make an *ansatz* for a purely radial field, $\boldsymbol{E}(\boldsymbol{x}) = E(r)\,\hat{e}_r$, and integrate Gauss's law over a spherical volume.

Explicitly calculate and plot the field $\boldsymbol{E}(\boldsymbol{x})$ and the potential $\varphi(\boldsymbol{x})$ for

b) a homogeneously charged sphere

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if} \quad r \leq r_0 \\ 0 & \text{if} \quad r > r_0 \; . \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(\boldsymbol{x}) = \sigma_0 \, \delta(r - r_0) \; .$$

(8 points)

2.2.4. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \boldsymbol{\nabla}^2)\varphi(\boldsymbol{x}) = 4\pi\rho(\boldsymbol{x})$$

in terms of an integral.

hint: The answer is a generalization of Poisson's formula.

(3 points)

. . . **/over**

2.3.1. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r} Q + \frac{1}{r^3} \boldsymbol{x} \cdot \boldsymbol{d} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \ldots$$

with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

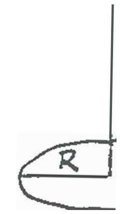
b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from part c) and calculate the quadrupole moments Q_{2m} as defined in ch.2 §3.5.

(10 points)



2.2.1 a) let the ring be in the z=0 plane.

$$\rho(\vec\xi) = \rho_0\, \delta(\xi_z)\, \delta(r-R)$$

in cylinder coordinates.

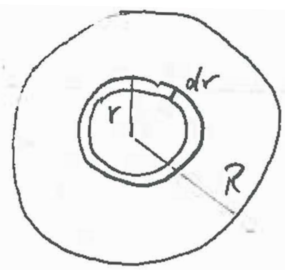
Total charge: $\int d\vec\xi\, \rho(\vec\xi) = 2\pi\rho_0 =: Q$

Poisson's formula: $\varphi(\vec x) = \int d\vec\xi\, \frac{\rho(\vec\xi)}{|\vec x-\vec\xi|}$

electric field: $\vec E = -\nabla\varphi = -\int d\vec\xi\, \rho(\vec\xi)\, \nabla\frac{1}{|\vec x-\vec\xi|} = \int d\vec\xi\, \rho(\vec\xi)\, \frac{\vec x-\vec\xi}{|\vec x-\vec\xi|^{3/2}}$

symmetry ~> $\vec E(\vec x=(0,0,z)) = E(z)\,\hat z$

1 ~> $E(z) = z\int d\vec\xi\, \frac{\rho(\vec\xi)}{|\vec x-\vec\xi|^{3/2}} = z\int_0^{2\pi} d\varphi\, \frac{\rho_0}{(z^2+R^2)^{3/2}} = \frac{Qz}{(z^2+R^2)^{3/2}}$



b) charge density: σ

-> charge on ring with radius r, thickness dr:

$$dQ = \sigma\, 2\pi r\, dr = \frac{Q}{\cancel{\pi} R^2}\, 2\cancel{\pi} r\, dr = \frac{2Q}{R^2}\, r\, dr \qquad Q=\pi R^2\sigma$$

c) ~> $E(z) = \int_0^R dr\, \frac{2Q}{R^2}\, r\, \frac{z}{(z^2+r^2)^{3/2}} = \frac{2Q}{R^2}\, z\, \frac{1}{2}\int_0^{R^2} \frac{dx}{(z^2+x)^{3/2}}$

1 $= -\frac{2Q}{R^2}\, \frac{1}{(1+x)^{1/2}}\Big|_0^{R^2/z^2} = \frac{2Q}{R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right) = 2\pi\sigma\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)$

$E(z\to\infty) = \frac{2Q}{R^2}\left(1-1+\frac{1}{2}\frac{R^2}{z^2}+O(R^4/z^4)\right) = Q/z^2 + O(z^{-4})$ field of point charge.

$\underline{E(z \to 0) = E(R \to \infty)} = \underline{2\pi\sigma}$

(1) An infinite sheet with constant charge density produces a field that's independent of z !

2.2.2 a) Consider Gauss's law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

and integrate over a sphere with volume V:

$$\int_V d\vec{x}\, \vec{\nabla}\cdot\vec{E} = \int_{\partial(V)} d\vec{S}\cdot\vec{E} = 4\pi \int_V d\vec{x}\, \rho$$

let $\rho(\vec{x})$ be spherically symmetric, $\rho(\vec{x}) = \rho(r)$, and make an

ansatz: $\boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$ $\quad \hat{e}_r = \vec{x}/|\vec{x}|$

$$4\pi r^2 E(r) = 4\pi\cdot 4\pi \int_0^r dr'\, r'^2 \rho(r')$$

1

$$\boxed{E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2 \rho(r')}$$

For the potential, consider $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$

spherical symmetry $\rightsquigarrow$ $\vec{\nabla}\varphi = \partial_r\varphi\, \hat{r}$

$\rightsquigarrow \quad E(r) = -\partial_r \varphi(r)$

$\rightsquigarrow \quad \varphi(r) = -\int_\infty^r dr'\, E(r')$ $\quad$ if we choose $\varphi(r=\infty) = 0$

$\rightsquigarrow \quad \varphi(r) = \int_r^\infty dr'\, E(r')$

1

b) $E(r) = \frac{4\pi}{r^2}\int_0^r dr'\, r'^2 \rho_0\, \Theta(r' < r_0)$

1st case: $r < r_0$ $\quad \underline{\underline{E(r)}} = \frac{4\pi\rho_0}{r^2}\int_0^r dx\, x^2 = \frac{4\pi\rho_0}{r^2}\frac{1}{3}r^3 = \frac{4\pi}{3}\rho_0 r$

$$= \frac{4\pi}{3} r_0^3 \rho_0 \frac{r}{r_0^3} = \frac{Qr}{r_0^3}$$

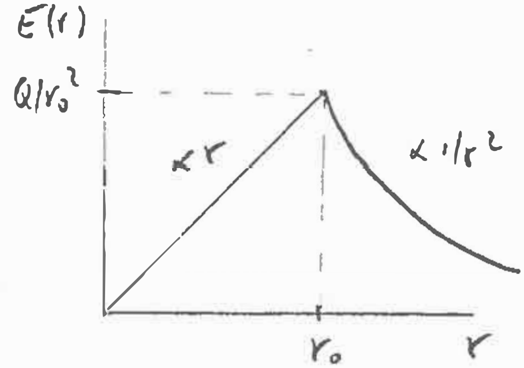
$Q = \frac{4\pi}{3} r_0^3 \rho_0$

= total charge

2nd case: $r > r_0$ $\quad E(r) = \frac{4\pi}{r^2}\int_0^{r_0} dr' r'^2 \rho_0 = \frac{4\pi}{r^2}\rho_0 \frac{1}{3} r_0^3 = Q/r^2$

$$\rightarrow \boxed{E(r) = \begin{cases} Qr/r_0^3 & \text{for } r \le r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$

$$\boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$$



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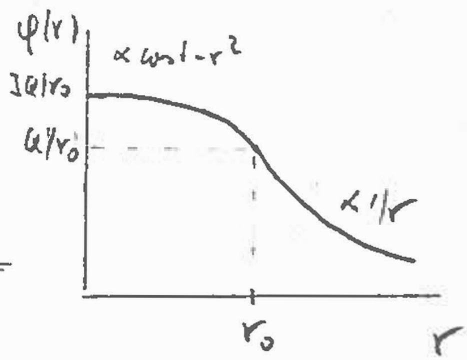
Now the potential:

1st case: $r < r_0$ $\quad \varphi(r) = \int_r^{r_0} dr' \frac{Qr'}{r_0^3} + \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r_0^3}\frac{1}{2}(r_0^2 - r^2) + \frac{Q}{r_0}$

$$= \frac{Q}{2r_0^3}(3r_0^2 - r^2)$$



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2nd case: $r > r_0$ $\quad \varphi(r) = \int_r^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r}$

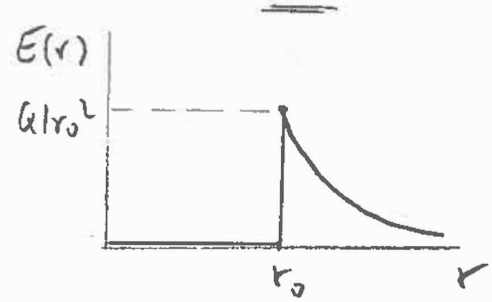
$$\boxed{\varphi(r) = \begin{cases} \frac{Q}{2r_0^3}(3r_0^2 - r^2) & \text{for } r < r_0 \\ Q/r & \text{for } r \ge r_0 \end{cases}}$$

1

c) electric field: $r < r_0$ $\quad E(r) = 0$

$r > r_0$ $\quad E(r) = \frac{4\pi}{r^2}\sigma_0 r_0^2 = \frac{Q}{r^2}$

$$\boxed{E(r) = \begin{cases} 0 & \text{for } r < r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$



with $Q = 4\pi r_0^2 \sigma_0$ = total charge

1

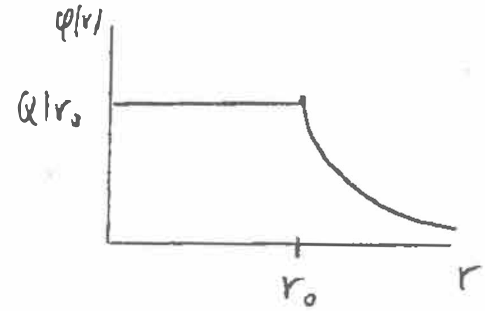
For $r > r_0$, $E(r)$ is the same as for the homogeneous sphere!

potential : $r < r_0$ $\quad \varphi(r) = \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = Q/r_0$

$r > r_0$ $\quad \varphi(r) = \int_{r}^{\infty} dr' \frac{Q}{r'^2} = Q/r$

1

$$\boxed{\varphi(r) = \begin{cases} Q/r_0 & \text{for } r < r_0 \\ Q/r & \text{for } r > r_0 \end{cases}}$$



2.2.4:) Helmholtz eq: $(\vec{\kappa}^2 - \vec{\nabla}^2)\varphi(\vec{x}) = 4\pi\rho(\vec{x})$

Fourier trafo as in ch 3 § 2:

1 $$(\vec{\kappa}^2 + \vec{k}^2)\hat{\varphi}(\vec{k}) = 4\pi\hat{\rho}(\vec{k})$$

$$\to \hat{\varphi}(\vec{k}) = \frac{4\pi}{\kappa^2 + \vec{k}^2}\hat{\rho}(\vec{k})$$

$$\to \varphi(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}} \frac{4\pi}{\kappa^2 + \vec{k}^2}\hat{\rho}(\vec{k})$$

$$= \int d\vec{y}\; v_{sc}(\vec{x} - \vec{y})\,\rho(\vec{y})$$ by the convolution theorem, PHYS 610 ch 2 § 7.1

1 where $v_{sc}(\vec{x})$ is the Fourier back trafo of the screened Coulomb potential

$$\hat{v}_{sc}(\vec{k}) = \frac{4\pi}{\kappa^2 + \vec{k}^2}$$

610 Problem 27 b) $\to$ $v_{sc}(\vec{x}) = \frac{1}{r}e^{-\kappa r}$ with $r = |\vec{x}|$

$$\to \boxed{\varphi(\vec{x}) = \int d\vec{y}\; \frac{e^{-\kappa|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\,\rho(\vec{y})}$$

1 For $\kappa = 0$ we recover Poisson's formula

2.3.1.) a) $\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{r}\left(1 - 2\frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{y^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1 + \frac{\vec{x}\cdot\vec{y}}{r^2} - \frac{1}{2}\frac{y^2}{r^2} + \frac{3}{2}\frac{(\vec{x}\cdot\vec{y})^2}{r^4} + \dots\right)$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{3}{2}x_i x_j y_i y_j \frac{1}{r^4} - \frac{1}{2}y^2\delta_{ij}x_i x_j \frac{1}{r^4} + \dots\right]$$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{1}{2}x_i x_j (3y_i y_j - \delta_{ij}y^2)\frac{1}{r^4} + \dots\right]$$

$$\rightsquigarrow \varphi(\vec{x}) = \int d\vec{y}\,\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} = \frac{1}{r}\int d\vec{y}\,\rho(\vec{y}) + \frac{1}{r^3}\vec{x}\cdot\int d\vec{y}\,\vec{y}\rho(\vec{y})$$

$$+ \frac{1}{2}\frac{1}{r^5}x_i x_j \int d\vec{y}\,(3y_i y_j - \delta_{ij}y^2)\rho(\vec{y}) + \dots$$

$$= \frac{1}{r}Q + \frac{1}{r^3}\vec{x}\cdot\vec{d} + \frac{1}{r^5}\sum_{ij} x_i x_j Q_{ij} + O(1/r^4)$$

where $Q = \int d\vec{y}\,\rho(\vec{y})$ monopole moment

$\vec{d} = \int d\vec{y}\,\vec{y}\rho(\vec{y})$ dipole moment

$Q_{ij} = \frac{1}{2}\int d\vec{y}\,(3y_i y_j - \delta_{ij}y^2)\rho(\vec{y})$ quadrupole moment

1

b) $\rho'(\vec{y}) = \rho(\vec{y}-\vec{a})$

$$\rightsquigarrow Q'_{ij} = \frac{1}{2}\int d\vec{y}\,(3y_i y_j - \delta_{ij}y^2)\rho'(\vec{y})$$

$$= \frac{1}{2}\int d\vec{y}\left[3(y_i+a_i)(y_j+a_j) - \delta_{ij}(\vec{y}+\vec{a})^2\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{1}{2}\int d\vec{y}\left[3a_i y_j + 3a_j y_i + 3a_i a_j - \delta_{ij}(2\vec{a}\cdot\vec{y} + \vec{a}^2)\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{3}{2}a_i d_j + \frac{3}{2}a_j d_i + \frac{3}{2}a_i a_j Q - \delta_{ij}\vec{a}\cdot\vec{d} - \delta_{ij}\frac{1}{2}\vec{a}^2 Q$$

$$= Q_{ij} \quad \text{if} \quad \vec{d} = Q = 0$$

1

c) ellipsoid: $x^2/a^2+y^2/b^2+z^2/c^2 \le 1$

$$\rightarrow\; Q_{ij} = \frac{1}{2}\int d\vec{x}\,\left(3x_i x_j - \delta_{ij}\vec{x}^2\right)\,\Theta\!\left(x^2/a^2+y^2/b^2+z^2/c^2 \le 1\right)\,\rho$$

where $\vec{x} = (x, y, z)$ and ρ = charge density

(1) symmetry $\rightarrow$ $\underline{Q_{ij} = 0 \text{ unless } i=j}$

$$\underline{\underline{Q_{11}}} = \frac{\rho}{2}\int dx\,dy\,dz\,\left(2x^2-y^2-z^2\right)\,\Theta\!\left(x^2/a^2+y^2/b^2+z^2/c^2 \le 1\right)$$

$$= \frac{1}{2}\rho abc \int dx\,dy\,dz\,\left(2a^2x^2-b^2y^2-c^2z^2\right)\,\Theta\!\left(x^2+y^2+z^2 \le 1\right)$$

$$= \frac{1}{2}\rho abc\Big[2a^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\; r^2\sin^2\vartheta\cos^2\varphi$$
$$- b^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\; r^2\sin^2\vartheta\sin^2\varphi$$
$$- c^2\int_0^1 dr\,r^2\int_{-1}^{1} d\eta\int_0^{2\pi} d\varphi\; r^2\cos^2\vartheta\Big]$$

$$= \frac{1}{2}\rho abc\left[2a^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - b^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - c^2 2\pi\,\frac{2}{3}\,\frac{1}{5}\right]$$

$$= \frac{1}{2}\rho abc\,\frac{\pi}{5}\left[\frac{8}{3}a^2 - \frac{4}{3}b^2 - \frac{4}{3}c^2\right]$$

$$= \frac{4\pi}{3}abc\rho\,\frac{1}{10}\left[2a^2-b^2-c^2\right]$$

(1) $$= \underline{Q\,\frac{1}{10}\left(2a^2-b^2-c^2\right)}$$ with $\underline{Q = \frac{4\pi}{3}\rho abc}$ = total charge

$\underline{\underline{Q_{22}}} = \underline{Q\,\frac{1}{10}\left(2b^2-a^2-c^2\right)}$ by symmetry

$\underline{\underline{Q_{33}}} = \underline{Q\,\frac{1}{10}\left(2c^2-b^2-a^2\right)}$ by symmetry

(1) check: $\underline{Q_{11}+Q_{22}+Q_{33} = 0}$ ✓

d) As a real symmetric tensor, Q_{ij} can always be diagonalized

→ The most general form of Q_{ij} in its principal axes system is

$$Q_{ij} = \begin{pmatrix} q_+ + q_- & 0 & 0 \\ 0 & q_+ - q_- & 0 \\ 0 & 0 & -2q_+ \end{pmatrix}$$

where

$$\underline{\underline{q_-}} = \frac{1}{2}\left(Q_{11} - Q_{22}\right) = \frac{1}{2}\int d\vec{x}\, \rho(\vec{x})\, \left[2x^2 - y^2 - z^2 - 2y^2 + x^2 + z^2\right]$$

$$= \underline{\frac{3}{2}\int d\vec{x}\, \rho(\vec{x})\, (x^2 - y^2)}$$

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cylinder coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$, $x^2 - y^2 = r^2(\cos^2\varphi - \sin^2\varphi) = r^2\cos 2\varphi$

$$\to q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r \int dz\, \rho(r,\varphi,z)\, r^2 \cos 2\varphi$$

Now let $\rho(r,\varphi,z) = \rho(r,\varphi+\alpha,z)$

$$\to q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi+\alpha,z) \cos 2\varphi$$

$$= \frac{3}{2}\int_\alpha^{2\pi+\alpha} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z) \cos 2(\varphi-\alpha)$$

$$= \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z) \left(\cos 2\varphi \cos 2\alpha + \sin 2\varphi \sin 2\alpha\right)$$

$\boxed{r^2 \sin 2\varphi = 2r^2 \sin\varphi\cos\varphi = xy \to \text{the second term is} \propto Q_{12} = 0}$

$$= \cos 2\alpha \cdot q_- \quad \to \quad \underline{\underline{q_- = 0}} \text{ since } \alpha \neq \pi$$

1

part c) with $a = b$ → $Q_{11} = Q_{22} = \frac{Q}{10}(a^2 - c^2)$ ✓

e) $\underline{\underline{Q_{20}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{2}(3\xi^2-1)$

$= \frac{1}{2} \int d\vec{x}\, \rho(\vec{x})\,(3z^2 - r^2) = \underline{\underline{D_{33}}}$

(1)

$\underline{\underline{Q_{2,\pm 1}}} \propto \int d\Omega\, \underbrace{\underbrace{\rho(r,\Omega)}_{\text{even}}\ \underbrace{P_2^{\pm 1}(\xi)}_{\text{odd}}}_{\text{fct. of } \xi} = \underline{\underline{0}}$ by symmetry

(1)

$\underline{\underline{Q_{22}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\frac{5}{4\pi}}}\, \frac{1}{\sqrt{4!}}\, e^{2i\varphi}\, 3(1-\xi^2)$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (1-\xi^2)(\cos 2\varphi + i \sin 2\varphi)$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 \sin^2\vartheta\, (\cos^2\varphi - \sin^2\varphi) + \frac{3i}{\sqrt{24}} \int d\vec{x}\, \underbrace{\underbrace{\rho(\vec{x})}_{\text{even}}\, r^2 \sin^2\vartheta\, \underbrace{\sin 2\varphi}_{\text{odd}}}_{\text{fct. of } \varphi\ \to 0}$

$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (\sin^2\vartheta \cos^2\varphi - \sin^2\vartheta \sin^2\varphi)$

$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \\ y &= r\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\} \rightsquigarrow x^2 - y^2 = r^2 \sin^2\vartheta\,(\cos^2\varphi - \sin^2\varphi)$$

$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x})\,(x^2 - y^2)$

$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x}) \left[(2x^2 - y^2 - z^2) - (2y^2 - x^2 - z^2)\right] \frac{1}{3}$

$= \underline{\underline{\frac{1}{\sqrt{6}} (D_{11} - D_{22})}}$

$\underline{\underline{Q_{2,-2}}} = \cancel{\sqrt{\frac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\frac{5}{4\pi}}}\, \sqrt{4!}\, e^{-2i\varphi}\, \frac{1}{8}(1-\xi^2) = \underline{\underline{Q_{22}}}$

(1)