Math 634, Homework #8
NOT TO BE HANDED IN

1. At this point you should be able to easily compute the homology groups of any 2-dimensional space. Just to make sure, let T be a torus and let $A, B \hookrightarrow T$ be two parallel circles. Let X be the space obtained from T by collapsing all of A to a point, and all of B to a different point. Compute the homology groups of X.

2. Now let us start to consider 3-dimensional manifolds. The ones you know at this point are
$$S^3, \quad \mathbb{R}P^3, \quad S^2 \times S^1, \quad T^g \times S^1, \quad (\mathbb{R}P^2 \# \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2) \times S^1$$
where T^g is the genus g torus. Make a table showing the homology groups with $\mathbb{Z}$-coefficients, with $\mathbb{Q}$-coefficients, and with $\mathbb{Z}/2$-coefficients for each of these spaces.

3. Recall that we made 2-dimensional manifolds by starting with a polygon in $\mathbb{R}^2$ and identifying boundary edges in pairs. One can make 3-dimensional manifolds in the same way, by starting with a polyhedron in $\mathbb{R}^3$ and identifying boundary faces in pairs. Poincaré explored several examples like this in his first papers on algebraic topology.

   Let X be the quotient of the cube $I \times I \times I$ in which one identifies each face with its opposite face via a clockwise 90 degree rotation. Compute the homology groups of X and prove that this 3-manifold is different from all the ones listed in problem #2. (Suggestion: Write down the cellular chain complex).

4. Let p and q be relatively prime, positive integers. Consider the 3-disk $D^3 \subseteq \mathbb{R}^3$, and regard it as the space
$$D^3 = \{(z, r) |\, z \in \mathbb{C}, r \in \mathbb{R}, |z|^2 + r^2 \leq 1\}.$$
   Let $\zeta = e^{2\pi i/p}$, and let $L(p, q)$ be the quotient space of X where one identifies
$$(z, r) \sim (\zeta^q z, -r)$$
   if $r \leq 0$ and $|z|^2 + r^2 = 1$. Note that $L(p, q)$ is a 3-manifold (each point on the lower hemisphere of ∂D^3, strictly below the equator, is identified with exactly one point above the equator, whereas each point on the equator is identified with $p - 1$ other points). The space $L(p, q)$ is called a *lens space*. Note that $L(2, 1) = \mathbb{R}P^3$.

   Let $p\colon D^3 \to L(p, q)$ be the quotient map. Let $X_0 = \{p(1, 0)\}$, let $X_1 = \{p(z, 0) |\, z \in \mathbb{C}, |z|^2 = 1\}$, and let $X_2 = \{p(z, r) \mid |z|^2 + r^2 = 1\}$. Convince yourself that
$$\emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 = L(p, q)$$
   is a CW-structure on $L(p, q)$. Use this to compute $H_*(L(p, q))$. Again, prove that for $p \neq 2$ these manifolds do not coincide with anything seen in the problems #2 and #3.

5. (Challenge question) Let D be the solid dodecahedron, homeomorphic to D^3. Let $X = D/\sim$, where one identifies each face of D with its opposite face via the smallest clockwise rotation that is possible. Compute the homology groups of X. [Based on homology, it will look like X is the same as another 3-manifold that we have already seen; next quarter we will prove that this is not the case.]