MA 266 FALL 99  PRACTICE PROBLEMS FOR FINAL EXAM

Table 6.21 Elementary Laplace Transforms will be included with the final exam.

1. Which of \( y_1(t) = t^4 \), \( y_2(t) = t^2 \), \( y_3(t) = t^3 \) are solutions of the differential equation \( t^2 y'' - 3ty' + 3y = 0 \)?

2. Use the given direction field to sketch the solution of the corresponding initial value problem with \( y(0) = 1 \). Determine (approximately) the interval in which the solution is valid.

3. What is the largest interval for which a unique solution of the initial value problem
\[
t^2 y' + \frac{1}{1 + t} y = \frac{1}{t - 1}, \quad y(1) = 0,
\]
is guaranteed?

4. Which of the following initial value problems are guaranteed to have a unique solution?
A. \( y' = (ty)^{1/3}, \quad y(0) = 0 \), \( y(1) = 1 \), \quad y(1) = 0,
B. \( y' = (ty)^{1/3}, \quad y(0) = 1 \), \( y(1) = 1 \),
C. \( y' = (ty)^{1/3}, \quad y(0) = 0 \), \( y(1) = 1 \).
D. \( y' = (ty)^{1/3}, \quad y(1) = 1 \).

5. Consider the differential equation \( \frac{dy}{dt} = -(y - 1)(y - 4)^2 / 10, \quad t \geq 0, \quad y \geq 0 \), where the graph of \( F(t) = -(y - 1)(y - 4)^2 / 10 \) is indicated below.

(a) What are the equilibrium solutions?
(b) Which equilibrium solutions are stable?
(c) For which open intervals of \( y \) is the graph of \( y(t) \) increasing?
(d) Sketch the graph of the solution of the differential equation for \( t \geq 0 \) with each of the initial values \( y(0) = 0 \), \( y(0) = 1 \), \( y(0) = 2 \), \( y(0) = 3 \), \( y(0) = 4 \), \( y(0) = 5 \).

6. Determine whether each of the following differential equations is separable, homogeneous, linear, or exact:
(a) \( x + 2y + (2x + y) \frac{dy}{dx} = 0 \)
(b) \( 2y + 1 + (x + 2) \frac{dy}{dx} = 0 \)

7. Find the explicit form of the general solution of the differential equation \( y' = y^2 - 1 \).

8. Find the explicit form of the solution of the initial value problem \( y' = y^2, \quad y(0) = 1 \). What is the largest open interval on which the solution is valid?

9. Find the general solution of the differential equation \( y' + \left( 1 + \frac{1}{t} \right) y = \frac{1}{t} \).

10. Find the solution of the initial value problem \( y' = \frac{2y + t^2}{t}, \quad y(1) = 2 \).

11. (a) Solve the initial value problem \( y' - y = 2e^{-t} - 2, \quad y(0) = a \).
(b) For which initial value \( a \) will the solution approach infinity as \( t \) approaches infinity?
(c) For which initial value \( a \) will the solution approach negative infinity as \( t \) approaches infinity?
(d) For which initial value \( a \) will the solution remain bounded as \( t \) approaches infinity?

12. Find an implicit form of the general solution of the differential equation \( y^2 + 1 + (2y + x) \frac{dy}{dx} = 0 \).

13. Consider the initial value problem \( y' = xy + y^2, \quad y(3) = -1 \).
(a) Is the solution increasing or decreasing near \( (x, y) = (3, -1) \)?
(b) Is the solution concave upward or downward near \( (x, y) = (3, -1) \)?
(c) Are the Euler tangent line method approximations of the solution near \( (x, y) = (3, -1) \) greater than or less than the solution?

14. Determine approximate values at \( t = 0.5 \) of the solution of the initial value problem \( y' = 3y + y, \quad y(0) = 1 \) by using the Euler tangent line method with \( h = 0.25 \).

15. Use the formula \( y = tu \) to express the differential equation \( \frac{dy}{dt} = \frac{t + y}{t - y} \) in terms of \( t, u \), and \( \frac{du}{dt} \).

16. Suppose \( y' \) is proportional to \( y \), \( y(0) = 2 \), and \( y(1) = 8 \).
(a) Find \( y \) in terms of \( t \).
(b) For what value of \( t \) does \( y(t) = 20 \)?

17. Find the general solution of the differential equations.
(a) \( y'' - 4y' + 3y = 0 \), (b) \( y'' - 4y' + 4y = 0 \), (c) \( y'' + 4y' + 8y = 0 \).

18. Find the solution of the initial value problem \( y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = 0, \quad y'(0) = 1 \).

19. Find the general solution of the differential equation \( y^{(0)} + By^{(0)} + Cy'' + Dy' + Ey = 0 \), if its corresponding characteristic equation is \( (r + 1)(r^2 - 2r + 5) = 0 \).
In Problems 20–22 find the general solution of the homogeneous differential equation in (a) and use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equations in (b) and (c).

20. (a) \( y'' + 9y = 0 \), \( y'' + 9y = te^{-3t} \), \( y'' + 9y = \cos(3t) \)
21. (a) \( y'' + 6y' + 5y = 0 \), \( y'' + 6y' + 5y = t^2e^{-t} \), \( y'' + 6y' + 5y = e^t + e^{-t} \)
22. (a) \( y'' + 6y' + 9y = 0 \), \( y'' + 6y' + 9y = te^{-3t} \), \( y'' + 6y' + 9y = \cos(3t) + t \)

23. (a) Find the general solution of the homogeneous differential equation \( y'' + 2y' + y' = 0 \), which has characteristic equation \( r^2 + 2r + 1 = 0 \).
(b) Use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equation \( y'' + 2y' + y' = t + t \cos t \). You do not need to solve for the values of the coefficients.

24. Find the solution of the initial value problem \( y'' + 5y' + 6y = 24e^t \), \( y(0) = 0 \), \( y'(0) = 0 \).

25. Find the general solution of the differential equation \( y'' - y' = 4t \).

26. The differential equation \( t^2y'' - ty' + y = 0 \) has solution \( y_1(t) = t \).
Use the method of reduction of order to find a solution \( y_2(t) \) of the equation that is not a constant multiple of \( y_1 \).

27. The differential equation \( t^2y'' - 2ty' + 2y = 0 \) has solutions \( y_1 = t \) and \( y_2 = t^2 \). Use the
method of variation of parameters to find a solution of \( t^2y'' - 2ty' + 2y = 2t^2 \).

28. The functions \( y_1 = t^2 \) and \( y_2 = t^{-2} \) are solutions of the differential equation
\( t^2y'' + ty' - 4y = 0 \).
(a) Evaluate the Wronskian \( W(t^2, t^{-2}) \).
(b) Find the solution of the initial value problem \( t^2y'' + ty' - 4y = 0 \), \( y(1) = 1 \), \( y'(1) = 6 \).

29. (a) For what values of \( \omega \) will resonance occur, so the solution of the initial value problem \( y'' + 4u = 4 \cos(\omega t) \), \( u(0) = 0 \), \( u'(0) = 0 \), becomes unbounded as \( t \to \infty \)?
(b) For what value of \( \omega \) does the solution of the initial value problem \( y'' + 4u + 4u = 4 \cos(\omega t) \), \( u(0) = 0 \), \( u'(0) = 0 \), become unbounded as \( t \to \infty \)?
(c) For what positive value(s) of \( m \) will the solution of the initial value problem \( m^2u'' + 2mu + u = 0 \), \( u(0) = 0 \), \( u'(0) = 1 \) oscillate, so \( u(t) \) will periodically have value zero?

30. At time \( t = 0 \) a tank contains 40 ounces of salt mixed in 100 gallons of water. A solution that contains 4 oz of salt per gallon of solution is then pumped into the tank at a rate of 5 gal/min. The well-stirred mixture flows out of the tank at the rate of 3 gal/min. Set up and solve an initial value problem that gives the amount of salt in the tank after \( t \) minutes.

31. Use the definition of the Laplace transform as an improper integral to evaluate \( \mathcal{L}(u_1(t)e^{-t}) \).

32. Find the Laplace transform of the functions \( f(t) \).
(a) \( f(t) = \sin(3t) + \cos(3t) \),
(b) \( f(t) = e^{t} + \cos(2t) \),
(c) \( f(t) = t^2 - u_1(t)(t^2 - 1) \).

33. Find the inverse Laplace transform of the functions \( F(s) \).
(a) \( F(s) = \frac{s}{(s - 1)^2} \), \( F(s) = \frac{3}{s^2 - 2s - 3} \),
(c) \( F(s) = \frac{5e^{-2s}}{s^2 + 2s + 5} \).

34. (a) \( \mathcal{L} \left\{ \int_0^t \sin(2(t - r)) \cdot \cos(3r) \, dr \right\} = \)
(b) \( \mathcal{L}^{-1} \left\{ \frac{6}{s^4} \cdot \frac{1}{s^2 + 4} \right\} = \int_0^t \)

35. Find the Laplace transform of the function \( f(t) \), where the graph of \( f \) is given below.

36. Find the solution of the initial value problem \( y'' + y = F(t), \ y(0) = 0, \ y'(0) = 0, \ \text{where} \ F(t) = \begin{cases} t, & 0 \leq t < \pi, \\ \pi, & t \geq \pi \end{cases} \).

37. Find the solution of the initial value problem \( y'' + y = \delta(t - \pi), \ y(0) = 0, \ y'(0) = 1 \).

38. Express the initial value problem \( y'' + 2y' + (\sin t)y = \cos t \), \( y(0) = 1, \ y'(0) = -3 \), as a system of first order differential equations with initial conditions.

39. Find the general solution of each of the systems:
(a) \( \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \),
(b) \( \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \),
(c) \( \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \),
(d) \( \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \),
(e) \( \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \).
40. The homogeneous equation
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\
 x_2 \end{pmatrix}
\]
has general solution
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = C_1 \begin{pmatrix} 1 \\
0 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\
-1 \end{pmatrix} e^{-2t}.
\]
Find a particular solution of the nonhomogeneous equation
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\
 x_2 \end{pmatrix} + \begin{pmatrix} 3e^{-t} \\
0 \end{pmatrix}.
\]

41. Find the direction field that corresponds to each of the given systems of differential equations:
   (a) \[
   \begin{pmatrix}
   \dot{x}_1 \\
   \dot{x}_2
   \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix},
   \text{ solution } \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\
   0 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\
   0 \end{pmatrix} e^{-t},
   \]
   (b) \[
   \begin{pmatrix}
   \dot{x}_1 \\
   \dot{x}_2
   \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix},
   \text{ solution } \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\
   0 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 0 \\
   1 \end{pmatrix} e^t,
   \]
   (c) \[
   \begin{pmatrix}
   \dot{x}_1 \\
   \dot{x}_2
   \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix},
   \text{ solution } \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix} = C_1 \begin{pmatrix} \cos t \\
   \sin t \end{pmatrix} e^t + C_2 \begin{pmatrix} \sin t \\
   -\cos t \end{pmatrix} e^t,
   \]
   (d) \[
   \begin{pmatrix}
   \dot{x}_1 \\
   \dot{x}_2
   \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix},
   \text{ solution } \begin{pmatrix} x_1 \\
   x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\
   0 \end{pmatrix} e^t + C_2 \begin{pmatrix} 0 \\
   1 \end{pmatrix} e^t + \begin{pmatrix} 0 \\
   1 \end{pmatrix} e^t.
   \]

42. Tank 1 initially holds 50 gallons of brine with concentration 2 oz/gal and Tank 2 initially holds 25 gallons of pure water. Brine with concentration 4 oz/gal flows into Tank 1 at a rate of 6 gal/min and brine with concentration 3 oz/gal flows into Tank 2 at a rate of 4 gal/min. The well-mixed solution in Tank 1 flows out of the system at a rate of 3 gal/min and the solution in Tank 2 flows out of the system at a rate of 7 gal/min. Also, the solution in Tank 1 flows into Tank 2 at a rate of 5 gal/min while the solution in Tank 2 flows into Tank 1 at a rate of 2 gal/min. SET UP an initial value problem that gives the amounts of salt in Tank 1 and Tank 2 at time \( t \). DO NOT SOLVE THE INITIAL VALUE PROBLEM!

ANSWERS:

1. \( y_1(t) = t \) and \( y_2(t) = t^3 \)

2. The solution is valid for approximately \(-1 < t < 1\).

3. \( 0 < t < 3 \)

4. B and D.
(The function \( f(t, y) = (ty)^{1/3} \) is continuous for all \((t, y)\); \( \frac{\partial f}{\partial y} \) is continuous for \( y \neq 0 \).)

5. (a) \( y = 1 \) and \( y = 4 \)
   (b) \( y = 1 \)
   (c) \( 0 < y < 1 \)

6. (a) homogeneous and exact, (b) separable and linear

7. \( y = \frac{1 + Ce^{2t}}{1 - Ce^{2t}}, y = \pm 1 \)

8. \( y = \frac{1}{\sqrt{1 - 2t}}, t < \frac{1}{2} \)

9. \( y = \frac{1}{t} + \frac{C}{te^t} \)

10. \( y = t^3 \ln t + 2t^2 \)

11. (a) \( y = 2 - e^{-t} + (a - 1)e^t \), (b) \( a > 1 \), (c) \( a < 1 \), (d) \( a = 1 \).

12. \( x^2 + y = C \)

13. (a) decreasing, (b) concave downward, (c) The Euler tangent line approximations are greater than the solution near \((x, y) = (3, -1)\).
14. 1.75
15. \( \frac{dv}{dt} = \frac{1 + v^2}{1 - v} \)
16. (a) \( y = 2e^{(ln4)t} = 2 \cdot 4^t \), (b) \( t = \frac{ln10}{ln4} \)
17. (a) \( y = C_1 e^t + C_2 e^{2t} \)
(b) \( y = C_1 e^{2t} + C_2 e^{4t} \)
(c) \( y = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t \)
18. \( y = 3 - e^t + te^t \)
19. \( y = C_1 e^{-t} + C_2 e^t \cos(2t) + C_3 e^t \sin(2t) + C_4 e^t \cos(2t) + C_4 e^t \sin(2t) \)
20. (a) \( y = C_1 \cos(3t) + C_2 \sin(3t) \), (b) \( y = (At + B)e^{-3t} \), (c) \( y = t(A \cos(3t) + B \sin(3t)) \)
21. (a) \( y = C_1 e^{-3t} + C_2 e^{-3t} \), (b) \( y = t(At^2 + Bt + C)e^{-t} \), (c) \( y = Ae^t + Bte^{-t} \)
22. (a) \( y = C_1 e^{-3t} + C_2 e^{-3t} \), (b) \( y = t^2(At + B)e^{-3t} \), (c) \( y = A \cos(3t) + B \sin(3t) + Ct + D \)
23. (a) \( y = C_1 + C_2 \cos t + C_3 \sin t + C_4 t \cos t + C_5 t \sin t \),
(b) \( y = (At + B)t + P((Ct + D) \cos t + (Et + F) \sin t) \)
24. \( y = -8e^{-2t} + 6e^{-3t} + 2e^t \)
25. \( y = C_1 + C_2 e^t - 2t^2 - 4t \)
26. \( y_2 = t \ln |t| \)
(Any \( y_2 = At \ln |t| + Bt \), where \( A \) and \( B \) are fixed numbers and \( A \neq 0 \) is correct.)
27. \( y = C_1 t + C_2 t^2 + 2t \ln |t| \), \( C_1 \) and \( C_2 \) any real numbers
28. (a) \( W(t^2, t^{-2}) = -4t^{-4} \), (b) \( y = 2t^2 - t^{-2} \)
29. (a) \( \omega = \pm 2 \), (b) none, (c) \( m > 1 \)
30. \( \frac{dQ}{dt} = 4(5) - \left( \frac{Q}{100 + 2t} \right) \) \( (3) \), \( Q(0) = 40 \); \( Q(t) = 8(t + 50) - \frac{360(50)^{3/2}}{(t + 50)^{3/2}} \)
31. \( L \left( u_1(t)e^{-t} \right) = \int_0^1 e^{-(t+1)^{3/2}} \, dt = \lim_{A \to \infty} \int_1^A e^{-(s+1)^{3/2}} \, ds \)
\[ \lim_{A \to \infty} \left( \frac{e^{-(s+1)}}{s+1} \right) = \lim_{A \to \infty} \left( \frac{e^{-(s+1)}}{s+1} \right) = \lim_{A \to \infty} \left( \frac{e^{-(s+1)}}{s} \right) = e^{-(s+1)} \]
32. (a) \( \frac{3}{s^3 + 9} + \frac{s}{s^3 + 9} \), (b) \( \frac{1}{s - 1} + \frac{s - 1}{(s - 1)^3 + 4} \), (c) \( \frac{2}{s^3} - e^{-t} \left( \frac{2}{s^3} + \frac{2}{s^2} \right) \)
33. (a) \( f(t) = e^t + te^t \), (b) \( f(t) = \frac{3}{4} e^{4t} + \frac{4}{s} e^{-t} \)
(c) \( f(t) = u_1(t) \left( e^{t+1} \cos(2t - 2) - \frac{1}{2} e^{t+1} \sin(2t - 2) \right) \)
34. (a) \[ L \left\{ \int_0^t \sin(2t - r) \cdot \cos(3r) \, dr \right\} = \frac{2}{s^2 + 4} \cdot \frac{s}{s^2 + 9} \]
(b) \[ L^{-1} \left\{ \frac{6}{s^4 + 4} \right\} = \int_0^t (r - t)^3 \cos 2r \, dr = \int_0^t r^3 \cos 2(t - r) \, dr \]
35. \( 10e^{-4t} \) \( \frac{s}{s^2 + 4} + 5 \frac{e^{-6s}}{s} - 5 \frac{e^{-8s}}{s} \)
36. \( y = t - \sin t - u_r(t)(t - \pi - \sin(t - \pi)) = \begin{cases} t - \sin t, & 0 < t < \pi, \\ \pi - 2\sin t, & t \geq \pi. \end{cases} \)
37. \( y = \sin t + u_r(t)(\sin(t - \pi)) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & t \geq \pi. \end{cases} \)
38. \( x_1' = x_2, \ x_2' = \cos t - 2x_2 - (\sin t)x_1, \ x_1(0) = 1, \ x_2(0) = -3 \)
39. (a) \( \frac{x_1}{x_2} = C_1 \left( \frac{\cos t}{\sin t} \right) + C_2 \left( -\cos t \right) \)
(b) \( \left( \frac{x_1}{x_2} \right) = C_1 \left( \frac{1}{1} \right) e^t + C_2 \left( \frac{1}{0} \right) e^t \)
(c) \( \left( \frac{x_1}{x_2} \right) = C_1 \left( \frac{0}{1} \right) e^t + C_2 \left( \frac{0}{1} \right) e^t \)
(d) \( \left( \frac{x_1}{x_2} \right) = C_1 \left( \frac{1}{0} \right) e^t + C_2 \left( \frac{1}{2} \right) e^{-t} + \left( \frac{1}{2} \right) e^t \)
(e) \( \left( \frac{x_1}{x_2} \right) = C_1 \left( \frac{2}{1} \right) e^t + C_2 \left( \frac{1}{1} \right) e^{-t} + \left( \frac{2}{1} \right) e^t \)
40. \( \frac{x_1}{x_2} = \left( \frac{1}{-2} \right) e^{-t} + \left( \frac{1}{0} \right) \)
41. (a) A, (b) E, (c) F, (d) C
42. \( \left( \frac{x_1}{x_2} \right) = \begin{cases} 24 - 8x_1/50 + 2x_2/25, & 12 + 5x_1/50 - 9x_2/25, \\ 100, & 0 \end{cases} \)