1. General Theory

Let $f: [a, b] \rightarrow [a, b]$ or $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Start with a specific number $x_0$. Compute the “orbit” $x_0, f(x_0), f(f(x_0)), \ldots$. We will consider the following questions:

- Do these numbers approach a limit?
- If not, do these numbers approach a cyclic orbit $p_1 \rightarrow p_2 \rightarrow \ldots \rightarrow p_n \rightarrow p_1$?
- If not, do these numbers behave chaotically?
- Does the behavior of the orbit depend on the starting point $x_0$?
- If $f$ depends on a parameter, how does the orbit behavior change as this parameter varies?

These questions cannot be easily answered. But various techniques have evolved which give partial insight. Computer experiments provide additional information.

We have dealt with the following main examples:

- The quadratic family $f(x) = x^2 + c: \mathbb{R} \rightarrow \mathbb{R}$
- The doubling function $D(x)$ from page 24.
- The ‘tent’ function $T(x)$, for instance see p. 24, #2c. (A similar function is dealt with on p.80, #9-13, but the story there is somewhat different—we will return to that example later).
- The logistic equation $f(x) = \lambda(1 - x)x: \mathbb{R} \rightarrow \mathbb{R}$, $\lambda > 0$.

2. Theoretical Tools

We have developed the following theoretical tools:

- Search first for fixed points $x_0$. Such points satisfy $f(x_0) = x_0$ and can be found by solving this equation.
- Search next for orbits of higher degree. Points in period two orbits satisfy $f(f(x_0)) = x_0$ and can be found by solving this equation. Points in higher degree orbits satisfy similar equations, but the equations rapidly become unmanagable. However, the points can be found graphically given an accurate graph.
- Fixed points can be attractive or repulsive. To determine which is the case, test whether $|f'(x_0)| < 1$ or not.
- Cycles can also be attractive or repulsive. To determine whether a two-cycle $x_0 \rightarrow x_1 \rightarrow x_0$ is attractive, test whether $|f'(x_0) \cdot f'(x_1)| < 1$ (or equivalently, test whether $|(f^2)'(x_0)| < 1$). A similar test exists for cycles of higher degree.
- Use graphical analysis and/or phase portraits.

**Exercise 2.1.** State very precisely the theorem that $|f'(x_0)| < 1$ implies $x_0$ is attractive.

Recall that the theorem is proved by applying the mean value theorem to $x_0$ and a nearby point $x$. According to the mean value theorem, there is a $c$ between $x$ and $x_0$ satisfying

$$
\frac{f(x) - f(x_0)}{x - x_0} = f'(c)
$$
Since the derivative is smaller than one and \( f(x_0) = x_0 \), this gives \( |f(x) - x_0| < |x - x_0| \). So each iteration carries \( x \) closer to \( x_0 \).

**Exercise 2.2.** Fill in the subtle details of this proof.

**Exercise 2.3.** State very precisely the theorem that \( |f'(x_0)| > 1 \) implies \( x_0 \) is repulsive.

Let \( x_0, x_1 \) be a two-cycle. We say that this cycle is attractive if there is an \( \epsilon > 0 \) such that whenever \( |x - x_0| < \epsilon \), then

\[
\lim_{n \to \infty} f^{2n}(x) = x_0 \quad \text{and} \quad \lim_{n \to \infty} f^{2n+1}(x) = x_1.
\]

**Exercise 2.4.** Explain why period two points are attractive if \( |f'(x_0) \cdot f'(x_1)| < 1 \). (Your proof should reduce this back to the fixed point theorem previously proved.)

### 3. The Doubling Function

We understand the dynamical behavior here by representing numbers by their binary expansion. (This is an example of a ‘coordinate change’, which we will learn much more about later). In binary, we have \( D(0.a_1 a_2 \ldots) = 0.a_2 a_3 \ldots \) You should be able to create cycles of various lengths using these ideas, and count the number of 2-cycles, 3-cycles, 4-cycles.

**Exercise 3.1.** Explain the following points:
1. \( D \) has cycles of every possible length.
2. Every rational number is an eventually periodic point of \( D \).
3. Suppose I am thinking of a particular cycle of \( D \), an arbitrary number \( a \in [0, 1) \), and a positive number \( \epsilon > 0 \). Explain why there are points in \((a - \epsilon, a + \epsilon)\) whose orbit converges to the given cycle, and how you could construct such a point.

Make sure you can convert base \( N \) numbers which are eventually periodic into their rational expressions. Why do rational numbers have base \( N \) expansions which are eventually periodic?

### 4. The Quadratic Map

The quadratic map is the function \( f(x) = x^2 + c \).

**Exercise 4.1.** If \( 1/4 < c \), explain why there are no fixed points and all orbits go to infinity.

**Exercise 4.2.** If \( c < 1/4 \), show that there are exactly two fixed points. Show that the right hand fixed point is always repulsive. Show that the left hand fixed point is attractive exactly when \(-3/4 < c < 1/4\).

**Exercise 4.3.** Find the period two cycles for the quadratic map. In particular, show that two-cycles exist exactly when \( c < -3/4 \).

**Exercise 4.4.** Show that the two-cycle is attractive exactly when \(-\frac{5}{4} < x < -\frac{3}{4}\).

You should now have a fairly good understanding of the dynamics of \( x^2 + c \) for \(-5/4 < c \). Later in the course, we will study other values of \( c \).

**Exercise 4.5.** Explain how you could go about determining (approximately) the values of \( c \) where \( f(x) \) has a 5-cycle, given a functioning computer.
5. The Tent Map

This is the map $T: [0, 1] \to [0, 1]$ defined by the rule

$$T(x) = \begin{cases} 
2x & \text{if } x \leq \frac{1}{2} \\
2 - 2x & \text{if } \frac{1}{2} \leq x
\end{cases}$$

Notice that this map has two fixed points, both repulsive.

**Exercise 5.1.** Count the number of 2-cycles and 3-cycles for $T$. Explain why $T$ has cycles of arbitrarily long length. Are these cycles attractive or repulsive? Which points in $[0, 1]$ have orbits which don’t converge to a cycle? Explain the following theorem: given any $a \in [0, 1]$ and any $\epsilon > 0$, there are points in $(a - \epsilon, a + \epsilon)$ whose orbit does not converge to any cycle.

6. The Logistic Equation

Consider the function $f(x) = \lambda x(1 - x)$. We now perform the following experiment. For each $\lambda$ in $[0, 1]$, start with $x_0 = \frac{1}{2}$ and iterate $x_n = f(x_{n-1})$ one hundred times, throwing the results away. Then iterate an additional hundred times, plotting the resulting points on the vertical axis.

We obtain the following picture:

![Logistic Equation Graph](image)

**Exercise 6.1.** Explain everything that you now understand about this diagram. Why does it begin with a horizontal line, and then continue with a rising line? Why does the rising line split, and then split again? Determine the values of $\lambda$ where the horizontal line starts to rise, and where the rising line splits. Determine the equation for the rising line. What part of the diagram shows an attractive 2-cycle? Determine a formula for this 2-cycle. Determine the values of $\lambda$ where this 2-cycle is attractive.

7. Miscellaneous Graphical Analysis Exercises

**Exercise 7.1.** Using graphical analysis, analyze the dynamics of $x^2 + c$ for $-5/4 < c$. For instance, which points go to infinity? What happens when $c = 1/4$? When $-5/4 < c < -3/4$, most points go to a two-cycle, but some points go to a repulsive fixed point—explain.
Exercise 7.2. Below is a picture of \( x, f(x) = x^2 + c \), and \( f(f(f(x))) \) for \( c = -1.78 \). Using this picture, explain why \( f \) has exactly two 3-cycles. From left to right, label these 3-cycle points a, b, c, d, e, f. Using graphical analysis, determine precisely how the three-cycles behave. For example, is \( a \to b \to c \to a \) a three-cycle?

Exercise 7.3. If you did the previous exercise correctly, you will have discovered that the slope of \( f(f(f(x))) \) is the same at each of the three points of a three cycle. (This will work with both of the three cycles in question.) Explain why this is a universal theorem.

8. Graphical Analysis

We end this handout sheet with three isolated pictures of graphical analysis.

Below is the analysis of a three cycle at the very moment that three cycles are first created for the family \( x^2 + c \).