PÓLYA'S THEOREM ON RANDOM WALKS VIA PÓLYA'S URN

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1. Introduction

Suppose a particle performs a simple random walk on the vertices of $\mathbb{Z}^d$, at each step moving to one of the $2d$ neighbors of the current vertex, chosen uniformly at random. The long-term behavior of the walk depends on the dimension d, as first proven in 1921 by G. Pólya:

Theorem 1 (Pólya [17])**.** *Consider the simple random walk on $\mathbb{Z}^d$. If $d \leq 2$, then with probability 1, the walk returns to its starting position. If $d \geq 3$, then with positive probability, the walk never returns to its starting position.*

This is Pólya's theorem referred to in the title. Theorem 1 motivates the definition of a *transient* graph. Let $\mathcal{G}$ be a graph with vertex set V and edge set E. We will often write $x \sim y$ to mean that $\{x, y\} \in E$. The ***degree*** of a vertex v, denoted $\deg(v)$, is the number of edges containing v. A graph is ***locally finite*** if all vertices have finite degree. We assume throughout this paper that all graphs are connected and locally finite. The ***simple random walk*** on $\mathcal{G}$ moves at each unit of time by selecting a vertex uniformly at random among those adjacent to the walk's current location. The graph $\mathcal{G}$ is ***transient*** if there is a positive probability that the simple random walk on $\mathcal{G}$ never returns to its starting position. The graph $\mathcal{G}$ is ***recurrent*** if it is not transient. (Since $\mathcal{G}$ is assumed to be connected, transience and recurrence do not depend on the starting vertex.) For more on transience and recurrence, see, for example, [14, 20, 9].

We will prove Theorem 1 by the method of flows, which we now describe. Let $\mathcal{G}$ be a connected graph with vertex set V and edge set E. An ***oriented*** edge is an ordered pair (x, y) such that $\{x, y\} \in E$. We will sometimes write $\vec{xy}$ to denote the oriented edge (x, y). A function θ defined on oriented edges is ***antisymmetric*** if $\theta(\vec{xy}) = -\theta(\vec{yx})$ for all oriented edges $\vec{xy}$. For a function θ defined on oriented edges, the

divergence of θ at v is defined as

$$(\operatorname{div}\theta)(v) := \sum_{w\,:\,w\sim v} \theta(\vec{vw}).$$

If $\mathcal{G}$ is infinite, a ***flow*** θ on $\mathcal{G}$ from vertex $a \in V$ is a function defined on oriented edges which is antisymmetric and satisfies ***Kirchoff's node law***,

$$(\operatorname{div}\theta)(v) = 0, \tag{1.1}$$

at all $v \neq a$. The above condition is that "flow in equals flow out" at v. The ***strength*** of a flow θ is defined as $\|\theta\| := (\operatorname{div}\theta)(a)$, the amount flowing from a. A ***unit flow*** has strength equal to 1. The ***energy*** $\mathcal{E}(\theta)$ of a flow θ equals $\sum_{e\in E}\theta(e)^2$. (Note the sum is over unoriented edges; although θ is defined for oriented edges, since θ is antisymmetric, $\theta(e)^2$ is well-defined by taking either orientation of e.) Any flow with positive strength and finite energy can be rescaled to yield a unit flow of finite energy.

The following theorem, due to T. Lyons, gives a sharp criterion for transience of a graph in terms of flows:

Theorem 2 (T. Lyons [11])**.** *A graph $\mathcal{G}$ is transient if and only if there exists a flow θ on $\mathcal{G}$ (from some vertex a) with $\|\theta\| > 0$ and $\mathcal{E}(\theta) < \infty$.*

While Theorem 2 is very well known, the published proofs we are aware of require either a rather lengthy introduction to electrical networks, as in [4], or familiarity with infinite-dimensional Hilbert space, as in [11]; while these are valuable tools, we have tried here to give a short and elementary route to Theorem 2 by presenting only the relevant part of the electrical network theory.

One nice immediate consequence of Theorem 2 is *Rayleigh's principle*: If $\mathcal{G}$ is recurrent, the same is true for any subgraph of $\mathcal{G}$ (since any flow on the subgraph is also a flow on $\mathcal{G}$). In particular, to prove Theorem 1, it is enough to prove that $\mathbb{Z}^2$ is recurrent and $\mathbb{Z}^3$ is transient.

Another contribution of Pólya to probability is the celebrated ***Pólya's urn*** [5, 18]. An urn initially contains one black and one white ball. At each unit of time, a ball is drawn at random, and replaced along with an additional ball of the same color. This is a key example of a *reinforced* process; see the survey by Pemantle [15] for more on such processes.

Pólya's urn can be generalized to three colors, say red, white, and blue. Let R_t, W_t, and B_t be the number of red, white, and blue balls, respectively, in the urn at time t. A key property of this process is that at time t, the random vector (R_t, W_t, B_t) is uniformly distributed over the set

$$\{(x, y, z) : x \geq 1, y \geq 1, z \geq 1, x + y + z = t + 3\}.$$

This is verified easily by induction, but we provide another argument in Section 2.

While it seems quite certain that the connection between Theorem 1 and the urn process was unknown to Pólya, in fact Pólya's urn can be used in the proof of Pólya's theorem on lattice random walks! The goal of this note is to give a self-contained proof of Theorem 1, via the method of flows, making essential use of Pólya's urn process.

We will write $(X_0, X_1, X_2, \ldots)$ for the sequence of vertices visited by a random walk on a connected graph $\mathcal{G}$. If $X_0 = a$ with probability 1, then we denote probabilities and expectations by $\mathbf{P}_a$ and $\mathbf{E}_a$, respectively. If A is an event, we write $\mathbf{1}_A$ for the $\{0, 1\}$-valued random variable which equals 1 if and only if the event A occurs.

Let $N_a = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = a\}}$ be the number of visits to a of a random walk on a connected graph, when started from vertex a. (Note that $N_a \geq 1$ since $X_0 = a$.) It is easy to show (see, e.g., [14]) that the following criteria are all equivalent, and do not depend on the choice of a:

(i) $\mathbf{P}_a\{N_a = 1\} = 0$,
(ii) $\mathbf{P}_a\{N_a = \infty\} = 1$, and
(iii) $\mathbf{E}_a N_a = \sum_{n=0}^{\infty} \mathbf{P}_a\{X_n = a\} = \infty$.

Our proof of Theorem 1 shows that (i) holds for the lattice walks in dimension 2 and fails in dimension 3. The classical proof for random walks on $\mathbb{Z}^d$ establishes the estimate $c_d n^{-d/2} \leq \mathbf{P}_0\{X_{2n} = 0\} \leq C_d n^{-d/2}$, which implies that the sum in (iii) converges if and only if $d \geq 3$. The advantage of the proof given here is that it is robust against small perturbation of the graph: if a few edges are removed or added to $\mathbb{Z}^d$, then the recurrence or transience property will not be altered.

The rest of the paper is organized as follows: we prove transience of $\mathbb{Z}^3$ (Theorem 2.2), assuming Theorem 2, in Section 2. In Section 3, we prove recurrence of $\mathbb{Z}^2$ (Theorem 3.2), again assuming Theorem 2. These two results together yield Theorem 1. We prove Theorem 2 In Section 4. We conclude in Section 5 with a discussion of transience of wedges.

2. The Pólya Flow and Transience in $\mathbb{Z}^3$

Consider the three-color version of Pólya's urn: Initially there are three balls, each a distinct color, say red, white, and blue. Each time a ball is drawn, the selected ball and a new one of the same color are placed back in the urn.

Lemma 2.1. *Let* (R_t, W_t, B_t) *be the number of red, white, and blue balls, respectively, in the urn after* t *draws. The vector* (R_t, W_t, B_t) *is uniformly distributed over*

$$V_t := \{(x, y, z) \,:\, x \geq 1,\, y \geq 1,\, z \geq 1,\, x + y + z = t + 3\}. \tag{2.1}$$

Proof. Let $U_{-1}, U_0, U_1, \ldots, U_n$ be independent and identically distributed random variables, each uniformly distributed over the interval $[0, 1]$. By symmetry, for $t = 1, \ldots, n$, each of the $(t+2)!$ possible orderings of the variables $U_{-1}, U_0, \ldots, U_t$ is equally likely. Define $V_1 := \min\{U_{-1}, U_0\}$, $V_2 := \max\{U_{-1}, U_0\}$, and for $t \geq 1$,

$$\begin{aligned}
X_t &:= |\{j \,:\, 1 \leq j \leq t \text{ and } 0 \leq U_j \leq V_1\}| + 1 \\
Y_t &:= |\{j \,:\, 1 \leq j \leq t \text{ and } V_1 < U_j \leq V_2\}| + 1 \\
Z_t &:= |\{j \,:\, 1 \leq j \leq t \text{ and } V_2 < U_j \leq 1\}| + 1.
\end{aligned}$$

That is, the first two variables U_{-1} and U_0 divide the interval $[0, 1]$ into three subintervals. The variable X_t equals one more than the number of the variables $\{U_1, \ldots, U_t\}$ falling in the left-most subinterval, Y_t is one more than the number falling in the middle interval, and Z_t is one more than the number falling in the right-most interval.

Set $(X_0, Y_0, Z_0) = (1, 1, 1)$. Observe that given the ordering of the variables $U_{-1}, U_0, \ldots, U_t$, the value of the triple (X_t, Y_t, Z_t) is determined.

We claim that (X_t, Y_t, Z_t) has the same distribution as (R_t, W_t, Z_t). Given the order of $(U_{-1}, U_0, \ldots, U_t)$, the rank of U_{t+1} among these values is uniformly distributed over all $t+3$ possible positions. Thus the conditional probabilities that U_{t+1} falls in the intervals $[0, V_1]$, $(V_1, V_2]$, and $(V_2, 1]$ are proportional to X_t, Y_t, and Z_t, respectively. We conclude that the transition probabilities for the triple (X_t, Y_t, Z_t) are the same as those for (R_t, W_t, B_t).

Using this equivalence, since the relative positions of $(U_{-1}, U_0, \ldots, U_t)$ form a random permutation of $\{1, \ldots, t+2\}$, it follows that (R_t, W_t, Z_t) is uniformly distributed over all $\binom{t+2}{2}$ ordered triples of positive integers summing to $t + 3$. □

Much more on urn processes can be found in [7], [16], and [12].

Theorem 2.2. *The simple random walk on* $\mathbb{Z}^3$ *is transient.*

Proof. Define a flow θ on $\mathbb{Z}^3$ from $(1, 1, 1)$ by orienting each edge in the octant $[1, \infty)^3$ in a north-east-up direction, and letting

$$\theta(\vec{e}) := \mathbf{P}\{\text{the urn process passes through } \vec{e}\}.$$

For $\vec{e}$ not in $[1,\infty)^3$, set $\theta(\vec{e}) := 0$.

Recall that V_t, defined in (2.1), is the set of vertices with positive coordinates t steps from $(1,1,1)$. There are $\binom{t+2}{2}$ such vertices, and by Lemma 2.1, the probability of the urn process visiting any one of them is $\binom{t+2}{2}^{-1}$. For $v \in V_t$, $t \geq 1$, the urn process visits v if and only if it traverses one of the oriented edges pointing to v, so we have

$$\sum_{u\,:\,u\to v} \theta(\vec{uv})^2 \leq \left[\sum_{u\,:\,u\to v} \theta(\vec{uv})\right]^2 = \mathbf{P}\{\text{the urn process visits } v\}^2 = \binom{t+2}{2}^{-2}.$$

(We have written $u \to v$ to indicate that there is an oriented edge pointing from u to v.) Therefore,

$$\sum_e \theta(\vec{e})^2 = \sum_{t=1}^{\infty} \sum_{v\in V_t} \sum_{u\,:\,u\to v} \theta(\vec{uv})^2 \leq \sum_{t=1}^{\infty} \binom{t+2}{2}\binom{t+2}{2}^{-2}. \tag{2.2}$$

The right-most sum above equals

$$\sum_{t=1}^{\infty} \frac{2}{(t+2)(t+1)} = 1.$$

The left-most sum in (2.2) is the energy of θ, so we have proven that $\mathcal{E}(\theta) \leq 1$. Theorem 2 completes the proof. □

The flow described in the proof of Theorem 2.2 can be found in [4], although those authors were not aware of the connection to Pólya's urn (Peter Doyle, personal communication).

3. Recurrence in $\mathbb{Z}^2$

Let $\mathcal{G}$ be an infinite graph. For a vertex a and a set of edges Π, let $V_{a,\Pi}$ be the union of $\{a\}$ and the set of all vertices v for which there exists a path from a to v not using an edge of Π. A set of edges Π is an ***edge-cutset*** separating a from ∞ if $V_{a,\Pi}$ is finite.

A technique due to Nash-Williams gives a sufficient condition for a graph to be recurrent.

Proposition 3.1 (Nash-Williams [13])**.** *For a graph $\mathcal{G}$, let $\{\Pi_k\}$ be disjoint finite edge-cutsets which separate the vertex a from infinity. If*

$$\sum_k |\Pi_k|^{-1} = \infty\,, \tag{3.1}$$

then the random walk on $\mathcal{G}$ is recurrent.

Let Π be an edge-cutset. For any flow θ from a, we have $(\operatorname{div}\theta)(a) = \|\theta\|$ and $(\operatorname{div}\theta)(v) = 0$ for $v \neq a$, whence

$$\|\theta\| = \sum_{v \in V_{a,\pi}} (\operatorname{div}\theta)(v) = \sum_{v \in V_{a,\Pi}} \sum_{w\,:\,w \sim v} \theta(\vec{vw}).$$

Since Π is an edge-cutset, the sum above is finite. Since any edge with both endpoints in $V_{a,\Pi}$ contributes two terms to the right-most sum which cancel each other, we have

$$\|\theta\| = \sum_{\substack{(v,w) \\ v \in V_{a,\Pi},\, w \notin V_{a,\Pi}}} \theta(\vec{vw}). \tag{3.2}$$

Any edge $\{v, w\}$ with $v \in V_{a,\Pi}$ and $w \notin V_{a,\Pi}$ must be in Π; therefore, the sum on the right-hand side of (3.2) is bounded above by $\sum_{e \in \Pi} |\theta(e)|$, showing that

$$\sum_{e \in \Pi} |\theta(e)| \geq \|\theta\|. \tag{3.3}$$

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Let θ be a unit flow from a. For any k, by the Cauchy-Schwarz inequality

$$|\Pi_k| \sum_{e \in \Pi_k} \theta(e)^2 \geq \left(\sum_{e \in \Pi_k} |\theta(e)| \right)^2.$$

By (3.3), the right-hand side is bounded below by $\|\theta\|^2 = 1$, because Π_k is a cutset and $\|\theta\| = 1$. Therefore

$$\sum_e \theta(e)^2 \geq \sum_k \sum_{e \in \Pi_k} \theta(e)^2 \geq \sum_k |\Pi_k|^{-1} = \infty.$$

By Theorem 2, we are done. □

Theorem 3.2. *The simple random walk on $\mathbb{Z}^2$ is recurrent.*

Proof. Let Π_k consist of those edges in $\mathbb{Z}^2$ between vertices at graph distance k and those at graph distance $k+1$ from 0. There are $8k + 4$ edges in Π_k, whence

$$\sum_{k=1}^{\infty} |\Pi_k|^{-1} \geq \sum_{k=1}^{\infty} \frac{1}{8k+4} = \infty.$$

Proposition 3.1 finishes the proof. □

4. Flows and Escape Probabilities

In this section, we will prove Theorem 2. For this, we will need to first develop a few tools.

4.1. **Flows on finite graphs and the current flow.** Let $\mathcal{G} = (V, E)$ be a *finite* graph. For vertices a and z, a ***flow from*** a ***to*** z is an antisymmetric function θ, defined on oriented edges, satisfying Kirchoff's node law at all $v \notin \{a, z\}$, and satisfying $(\operatorname{div}\theta)(a) \geq 0$.

We note that for any antisymmetric θ we have

$$\sum_{x \in V} (\operatorname{div}\theta)(x) = \sum_{x \in V} \sum_{\substack{y \in V \\ y \sim x}} \theta(\vec{xy}) = \sum_{\{x,y\} \in E} [\theta(\vec{xy}) + \theta(\vec{yx})] = 0. \tag{4.1}$$

The above implies that $(\operatorname{div}\theta)(a) = -(\operatorname{div}\theta)(z)$ for any flow θ.

The ***strength*** and ***energy*** of a flow θ from a to z are defined just as for flows on infinite graphs, namely, as $\|\theta\| := (\operatorname{div}\theta)(a)$ and $\mathcal{E}(\theta) := \sum_{e \in E} \theta(\vec{e})^2$, respectively. A ***unit flow*** from a to z is a flow with strength equal to 1.

An ***oriented cycle*** is a sequence of oriented edges $(\vec{e_1}, \ldots, \vec{e_r})$ such that $\vec{e_i} = (x_{i-1}, x_i)$ for vertices $(x_0, \ldots, x_r)$ with $x_r = x_0$. A flow from a to z satisfies the ***cycle law*** if $\sum_{i=1}^{r} \theta(\vec{e_i}) = 0$ for any oriented cycle $(\vec{e_1}, \ldots, \vec{e_r})$.

The unit flow from a to z satisfying the cycle law is unique:

Lemma 4.1. *There is at most one unit flow from a to z satisfying the cycle law.*

Proof. Suppose that θ and φ are two such flows. The function $f := \theta - \varphi$ satisfies the node law at all nodes and the cycle law. Suppose $f(\vec{e_1}) > 0$ for some oriented edge $\vec{e_1}$. By the node law, e_1 must lead to some oriented edge $\vec{e_2}$ with $f(\vec{e_2}) > 0$. Iterate this process to obtain a sequence of oriented edges on which f is strictly positive. Since the graph is finite, this sequence must eventually revisit a node. The resulting cycle violates the cycle law. □

We need to discuss the *Green's function* of a random walk before we construct a unit flow satisfying the cycle law.

For the simple random walk $(X_t)_{t=0}^{\infty}$ on the graph with vertex set V and edge set E, define for $B \subseteq V$

$$\tau_B := \inf\{t \geq 0 \,:\, X_t \in B\},$$
$$\tau_B^+ := \inf\{t > 0 \,:\, X_t \in B\}.$$

For a single site a, we write simply τ_a and τ_a^+ for $\tau_{\{a\}}$ and $\tau_{\{a\}}^+$, respectively.

The ***Green's function*** $G_z(x,y)$ is the expected number of visits of the random walk to y, when started at x, before hitting z:

$$G_z(x,y) = \mathbf{E}_x\left(\sum_{n=0}^{\tau_z-1} \mathbf{1}_{\{X_n=y\}}\right) = \sum_{n=0}^{\infty} \mathbf{P}_x\{X_n = y,\, \tau_z > n\}.$$

Since on a finite connected graph, $\mathbf{E}_x\tau_z < \infty$ for all x, z (see, e.g., [14]), we have $G_z(x,y) \leq \mathbf{E}_x\tau_z < \infty$ for finite connected graphs. Note that $G_z(x,z) = 0$ for $z \neq x$.

For any sequence of vertices $x_0, x_1, \ldots, x_{n-1}, x_n$ such that $\{x_i, x_{i-1}\} \in E$ and $x_0 = a$, $x_n = x$, it is easy to see that

$$\begin{aligned}\frac{1}{\deg(x)}&\mathbf{P}_a\{X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x\}\\ &= \frac{1}{\deg(a)}\mathbf{P}_x\{X_1 = x_{n-1}, X_2 = x_{n-1}, \ldots X_n = a\}.\end{aligned}$$

Summing the above identity over all such sequences with $x_i \neq z$ for $i = 1, \ldots, n$ shows that

$$\deg(x)^{-1}\mathbf{P}_a\{X_n = x,\, \tau_z > n\} = \deg(a)^{-1}\mathbf{P}_x\{X_n = a,\, \tau_z > n\},$$

and summing over $n \geq 0$ gives

$$\frac{G_z(a,x)}{\deg(x)} = \frac{G_z(x,a)}{\deg(a)}. \tag{4.2}$$

We are now able to give an explicit construction of the unique unit flow satisfying the cycle law, which we call the ***unit current flow from*** a ***to*** z and denote by I. Let $I(x,y)$ be the expected number of net traversals of the oriented edge (x,y) by the random walk started at a and stopped at z, where a traversal of the reversed edge (y,x) is counted with a negative sign. That is,

$$\begin{aligned}I(x,y) &:= \mathbf{E}_a\left(\sum_{n=0}^{\tau_z-1}[\mathbf{1}_{\{(X_n,X_{n+1})=(x,y)\}} - \mathbf{1}_{\{(X_n,X_{n+1})=(y,x)\}}]\right)\\ &= \sum_{n=0}^{\infty}\mathbf{P}_a\{X_n = x, X_{n+1} = y, \tau_z > n\}\\ &\quad - \sum_{n=0}^{\infty}\mathbf{P}_a\{X_n = y, X_{n+1} = y, \tau_z > n\}. \end{aligned} \tag{4.3}$$

Note that

$$\mathbf{P}_a\{X_n = x, X_{n+1} = y, \tau_z > n\} = \mathbf{P}_a\{X_n = x, \tau_z > n\}\mathbf{P}_x\{X_1 = y\}$$
$$= \frac{\mathbf{P}_a\{X_n = x, \tau_z > n\}}{\deg(x)},$$

and therefore

$$I(x,y) = \frac{G_z(a,x)}{\deg(x)} - \frac{G_z(a,y)}{\deg(y)} = \frac{G_z(x,a) - G_z(y,a)}{\deg(a)}, \tag{4.4}$$

where the last equality follows from (4.2).

It is easy to see from the above identity that I satisfies the cycle law. To see that I is a flow (i.e., it satisfies the node law at all vertices $x \notin \{a,z\}$), note that the number of entrances to such a vertex x must equal the number of exits from x. Finally, since the walk leaves a one more time than it enters a, we infer that I is a unit flow.

4.2. **The Laplacian.** For a function $f : V \to \mathbb{R}$, the ***Laplacian*** of f is the function Δf defined by

$$\Delta f(x) = f(x) - \frac{1}{\deg(x)} \sum_{y\,:\,y\sim x} f(y).$$

We say that f is ***harmonic*** at x if $\Delta f(x) = 0$. The value of a harmonic function at a vertex is the average of its values at neighboring vertices.

We will need the following two lemmas.

Lemma 4.2. *The function $g_{a,z}$ defined by $g_{a,z}(x) = G_z(x,a)$ is harmonic at all $x \notin \{a,z\}$ and has $\Delta g_{a,z}(a) = 1$.*

Proof. Let N_a be the (random) number of visits of a random walk to a (including at time 0) strictly before the first visit to z. Note that $g_{a,z}(x) = \mathbf{E}_x N_a$ and that

$$\mathbf{E}_x N_a = \frac{1}{\deg(x)} \sum_{y\,:\,y\sim x} \mathbf{E}_x(N_a \mid X_1 = y)$$
$$= \frac{1}{\deg(x)} \sum_{y\,:\,y\sim x} \mathbf{E}_y N_a + \mathbf{1}_{\{x=a\}}.$$

□

Lemma 4.3. *For any two functions φ and ψ defined on V,*

$$\frac{1}{2}\sum_{\substack{x,y\\ x\sim y}} [\varphi(y) - \varphi(x)][\psi(y) - \psi(x)] = \sum_x \varphi(x)\deg(x)\Delta\psi(x).$$

Proof. We have

$$\frac{1}{2}\sum_{\substack{x,y\\x\sim y}}[\varphi(y)-\varphi(x)][\psi(y)-\psi(x)] = \frac{1}{2}\sum_{\substack{x,y\\x\sim y}}\varphi(y)[\psi(y)-\psi(x)] + \frac{1}{2}\sum_{\substack{x,y\\x\sim y}}\varphi(x)[\psi(x)-\psi(y)].$$

By symmetry, the two terms on the right-hand side are the same, so their sum equals

$$\sum_x \varphi(x)\sum_{y\,:\,y\sim x}[\psi(x)-\psi(y)] = \sum_x \varphi(x)\deg(x)\Delta\psi(x).$$

□

4.3. **Proof of Theorem 2.** Before proving Theorem 2, which concerns *infinite* graphs, we first prove a related result about *finite* graphs:

Theorem 4.4. *For any finite connected graph,*

$$\deg(a)\mathbf{P}_a\{\tau_z < \tau_a^+\} = \frac{1}{\inf\{\mathcal{E}(\theta)\,:\,\theta \text{ a unit flow from } a \text{ to } z\}}.$$

Proof. Step 1: We show that the flow I defined in (4.3) is the unique flow minimizing $\mathcal{E}$.

Since the set of unit-strength flows with energy less than or equal to the energy of I is a closed and bounded subset of $\mathbb{R}^{|E|}$, by compactness there exists a flow θ minimizing $\mathcal{E}(\theta)$ subject to $\|\theta\| = 1$. By Lemma 4.1, to prove that the flow I is the unique minimizer, it is enough to verify that any flow θ of minimal energy satisfies the cycle law.

Let the edges $\vec{e_1},\ldots,\vec{e_n}$ form a cycle. Set $\gamma(\vec{e_i}) = 1$ for all $1 \le i \le n$ and set γ equal to zero on all other edges. Note that γ satisfies the node law, so it is a flow, but $\sum\gamma(\vec{e_i}) = n \ne 0$. For any $\varepsilon \in \mathbb{R}$, we have by energy minimality that

$$\begin{aligned}0 \le \mathcal{E}(\theta+\varepsilon\gamma) - \mathcal{E}(\theta) &= \sum_{i=1}^n \left[(\theta(\vec{e_i})+\varepsilon)^2 - \theta(\vec{e_i})^2\right]\\ &= 2\varepsilon\sum_{i=1}^n \theta(\vec{e_i}) + n\varepsilon^2.\end{aligned}$$

Dividing both sides by $\varepsilon > 0$ shows that

$$0 \le 2\sum_{i=1}^n \theta(\vec{e_i}) + n\varepsilon,$$

and letting $\varepsilon \downarrow 0$ shows that $0 \le \sum_{i=1}^n \theta(\vec{e_i})$. Similarly, dividing by $\varepsilon < 0$ and then letting $\varepsilon \uparrow 0$ shows that $0 \ge \sum_{i=1}^n \theta(\vec{e_i})$. Therefore, $\sum_{i=1}^n \theta(\vec{e_i}) = 0$, verifying that θ satisfies the cycle law.

Step 2: We show that

$$\mathcal{E}(I) = \frac{1}{\deg(a)\mathbf{P}_a\{\tau_z < \tau_a^+\}}. \tag{4.5}$$

From (4.4) and Lemma 4.3,

$$\begin{aligned}\sum_e I(e)^2 &= \frac{1}{\deg(a)^2}\frac{1}{2}\sum_{\substack{x,y\\ x\sim y}}[G_z(x,a) - G_z(y,a)]^2 \\ &= \frac{1}{\deg(a)^2}\sum_x G_z(x,a)\deg(x)\Delta g_{a,z}(x).\end{aligned}$$

From Lemma 4.2 and the fact that $G_z(z,a) = 0$, the right-hand side equals $\deg(a)^{-1}G_z(a,a)$. Note the $G_z(a,a)$ equals the expected number of visits to a, started from a, before hitting z. This is a geometric random variable with success probability $\mathbf{P}_a\{\tau_z < \tau_a^+\}$, whence $G_z(a,a) = 1/\mathbf{P}_a\{\tau_z < \tau_a^+\}$ and (4.5) is proven. □

Proof of Theorem 2. Let $\mathcal{G} = (V, E)$ be a transient connected graph. We will show that there exists a flow of finite energy. Let $d(v,w)$ be the ***graph distance*** between vertices v and w, equal to the length of the shortest path connecting v and w. Fix $a \in V$. Let $Z_n = \{v : d(a,v) \ge n\}$, and create a finite graph $\mathcal{G}_n$ by identifying all vertices in Z_n with a new vertex z_n, and removing all edges with both vertices in Z_n. Let I_n be the unit current flow on $\mathcal{G}_n$ from a to z_n. If $N_n(x,y)$ is the net number of traversals of the oriented edge (x,y) in $\mathcal{G}_n$ before hitting z_n, then $I_n(x,y) = \mathbf{E}_a N_n(x,y)$. With a minor abuse of notation, $N_n(x,y)$ is also the net number of traversals of the oriented edge (x,y) in $\mathcal{G}$ before hitting Z_n; with this interpretation, $N_n(x,y) \uparrow N(x,y)$, where $N(x,y)$ is the net number of traversals of (x,y) for the random walk in $\mathcal{G}$ started at a and run for infinite time. Since $\mathcal{G}$ is transient, $\mathbf{E}_a N(x,y) < \infty$, and by monotone convergence, $\mathbf{E}_a N_n(x,y) \uparrow \mathbf{E}_a N(x,y)$. That is, if $I(x,y) := \mathbf{E}_a N(x,y)$, then $I_n(x,y) \to I(x,y)$ as $n \to \infty$. The function $I(x,y)$ is clearly a unit flow from a on $\mathcal{G}$.

From (4.5),

$$\mathcal{E}(I_n) = \frac{1}{\deg(a)\mathbf{P}_a\{\tau_{z_n} < \tau_a^+\}}.$$

Note that

$$\mathbf{P}_a\{\tau_{z_n} < \tau_a^+\} = \mathbf{P}_a\{\tau_{Z_n} < \tau_a^+\},$$

where the probability on the left refers to the random walk on $\mathcal{G}_n$, and the probability on the right refers to the random walk on $\mathcal{G}$. Since the events $\{\tau_{Z_n} < \tau_a^+\}$ are decreasing with intersection $\{\tau_a^+ = \infty\}$, and $\mathbf{P}_a\{\tau_a^+ = \infty\} > 0$ by transience of $\mathcal{G}$, it follows that $\mathcal{E}(I_n) \le B < \infty$ for a finite constant B not depending on n.

For all m,

$$\sum_{\substack{x,y:x\sim y\\ d(x,a)<m,\, d(y,a)<m}} I(x,y)^2 = \lim_{n\to\infty} \sum_{\substack{x,y:x\sim y\\ d(x,a)<m,\, d(y,a)<m}} I_n(x,y)^2 \le \lim_{n\to\infty} \mathcal{E}(I_n) \le B < \infty.$$

Since this holds for all m, letting $m \to \infty$ shows that $\mathcal{E}(I) < \infty$.

For the converse, suppose that there exists a finite-energy unit flow θ from a in $\mathcal{G}$. Let θ_n be the restriction of θ to $\mathcal{G}_n$. (Note that $\mathcal{G}_n$ may contain multiple edges, as any edge (x,z) in G with $x \notin Z_n$ and $z \in Z_n$ yields an edge e from x to z_n with $\theta_n(e) = \theta(x,z)$.) Since $\mathcal{E}(\theta_n) \uparrow \mathcal{E}(\theta) < \infty$, we have

$$\mathbf{P}_a\{\tau_a^+ = \infty\} = \lim_{n\to\infty} \mathbf{P}_a\{\tau_{z_n} < \tau_a^+\} \ge \lim_{n\to\infty} \frac{1}{\deg(a)\mathcal{E}(\theta_n)} > 0, \tag{4.6}$$

so $\mathcal{G}$ is transient. □

5. Extensions and Further Reading

The connection between flows and escape probabilities forms a small part of the theory relating electrical networks to reversible Markov chains. See [4], [9], or [10] for much more on using electrical networks for calculating probabilities.

Dimension 2 is truly the critical dimension for recurrence. One way to show this is to consider a random walk in $\mathbb{Z}^3$ that usually makes a two-dimensional step to one of the four horizontal neighbors, but occasionally makes a three-dimensional step. If the number of three-dimensional steps among the first n steps grows like $(\log\log n)^{2+\epsilon}$ then the resulting process is transient; see [2]. More approachable by the methods discussed in this note are subgraphs of $\mathbb{Z}^3$ called ***wedges***. Let

$$W_f := \{(x,y,z) \in (\mathbb{Z}^+)^3 \,:\, y \le x,\ z \le f(x)\},$$

where f is an increasing function with $f(1) \ge 1$. We have

Theorem (T. Lyons [11])**.** *The wedge W_f is transient if and only if*

$$\sum_n [nf(n)]^{-1} < \infty.$$

Proof. Let Π_n be the cutset $\{\{(n,y,z),(n+1,y,z)\} : (n,y,z) \in W_f\}$. We have $|\Pi_n| = nf(n)$ and Proposition 3.1 implies that if $\sum_n 1/[nf(n)] = \infty$, then W_f is recurrent.

Now for the other implication. Consider the random spatial curve $\gamma(t) := (t, U_1 t, U_2 f(t))$, were U_1, U_2 are independent uniform $[0,1]$ random variables, and let Γ be the closest lattice path in W_f to γ. Define the flow θ by $\theta(\vec{e}) := \mathbf{P}\{\vec{e} \in \Gamma\}$. It can be shown that the energy of θ converges or diverges the same as the sum $\sum_n 1/[nf(n)]$, which proves that $\sum_n 1/[nf(n)] < \infty$ implies transience. □

The method of random paths has been used extensively to prove transience of super-critical percolation clusters in $\mathbb{Z}^d$ and in wedges; see [3], [6], and [1]. T. Lyons [11] used another flow to show transience of $\mathbb{Z}^3$.

Pólya described how he came to consider the problem of random walks:

> *At the hotel there lived also some students with whom I usually took my meals and had friendly relations. On a certain day one of them expected the visit of his fiancee, what I knew* [sic]*, but I did not foresee that he and his fiancee would also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly. I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case.* [19]

Thus, in fact, Pólya's motivation was to understand the number of *collisions* $\mathcal{C}$ of two independent random walks. For random walk on a lattice, this is equivalent to considering returns to zero for a new random walk. On a more general graph, however, the three properties (i)-(iii) on page 3 are not equivalent if $\mathcal{C}$ replaces N_a. It is true that for recurrent graphs of bounded degree, $\mathbf{E}\mathcal{C} = \infty$; nevertheless, this does not imply that $\mathbf{P}\{\mathcal{C} = \infty\} = 1$. Indeed, Krishnapur and Peres [8] show that for a certain subgraph of $\mathbb{Z}^2$ (which must be recurrent by Rayleigh's principle), the simple random walk satisfies $\mathbf{P}\{\mathcal{C} < \infty\} = 1$. The same authors also describe specific graphs for which it is not known whether the number of collisions of two independent walks is finite or infinite almost surely. It is an open problem to characterize graphs for which $\mathbf{P}\{C < \infty\} = 1$.

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